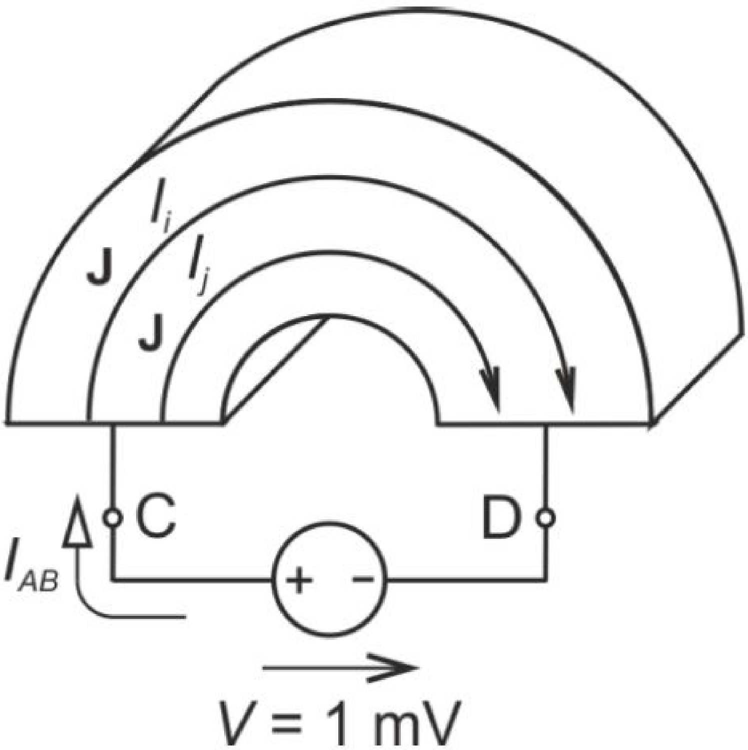
Electrical Circuits - Exercises 1-2.



István Gyurcsek

Electrical Circuits - Exercises 1-2

Pécs

2019

The Electrical Circuits - Exercises 1-2 course material was developed under the project EFOP 3.4.3-16-2016-00005 "Innovative university in a modern city: open-minded, value-driven and inclusive approach in a 21st century higher education model"

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Az Electrical Circuits - Exercises 1-2 tananyag az EFOP-3.4.3-16-2016-00005 azonosító számú,
„Korszerű egyetem a modern városban: Értékközpontúság, nyitottság és befogadó szemlélet egy 21. századi felsőoktatási modellben” című projekt keretében valósul meg.

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Értékközpontúság, nyitottság és befogadó szemlélet
egy 21. századi felsőoktatási modellben

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ELECTRICAL CIRCUITS - EXERCISES

DEPARTMENT OF ELECTRICAL NETWORKS
FACULTY OF ENGINEERING AND INFORMATION TECHNOLOGY
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Introduction and Scope of Work

Electromagnetic theory and electric circuit theory are the fundamental principles upon which all branches of electrical engineering are built. Many branches of electrical engineering like power, electric machines, control, electronics, communications and instrumentations are based on electric circuit theory. Therefore, basic electric circuit theory is not only one of the most important courses for students studying information technology and electrical engineering but it is also an excellent starting point for the fundamentals of any field of engineering.

Circuit theory is also valuable for students specialized in other branches of the physical sciences because circuits are good models for the study of energy systems in general, and because of the applied mathematics, physics and topology involved.

This textbook, which complements the theoretical summary of fundamentals in electrical engineering, is a short collection of calculation examples, to help students understand the basics of practical electricity, i.e. the basics of electric circuits. As both the theory and the practical experience are considerable, I found it a challenge to collect and summarize even the basic calculation examples in a short textbook. I focused on the most essential parts so many interesting and important practical sections of electricity are missing from this textbook. The topics were selected based on considerable educational experience and previously published textbooks and the content of this material has been structured accordingly.

I hope that this collection of practical examples covering the most important parts of the electrical circuit analysis will support you in understanding and practicing the basics of electric circuits and give you a sound background to build up deeper knowledge in practical engineering.

Any feedback regarding the structure and content of this material is welcome.

Finally, I would like to say thanks to everyone who helped me to create this book containing the most important parts of electrical circuit analysis essential to electric engineering.

I wish to say a special thanks to my colleagues for their help in the creation of this textbook - in this concise way, I couldn't have done such a good job of selecting the topics without their valuable support.

I also want to say a special thanks to the leadership of the University of Pécs, Faculty of Engineering and Information Technology for giving me the opportunity to write this book that I believe will be a useful tool for teaching our foreign and local students the basics of electricity.

Last but definitely not least, I would like to express my appreciation to my family. For a long period of time when I was preoccupied with preparing this material, I unintentionally tried their patience. I would like to extend a special thanks to them for their patience and understanding. They were extremely helpful, and I really appreciate it.

István Gyurcsek
Author

1. Stationary Electric Field

1.1 Differential and integral forms of Ohm's law

Find the electric current which flows through the aluminium rod shown in Figure 1.1 when 1mV is connected to its terminals. The length of rod is 10 cm, diameter is 5 mm and ρ_{Al} = 0.025 Ωmm^2/m. (The surfaces area, where the current enters/leaves the aluminium body, can be supposed to be equipotential.)

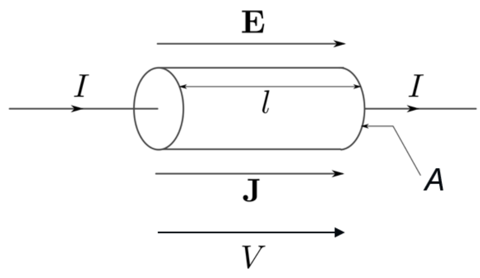

Figure 1.1 Current carrying aluminium rod

Solution

According to differential Ohm's law

$$\boldsymbol{J} = \sigma \cdot \boldsymbol{E} = \frac{1}{\rho} \cdot \boldsymbol{E} \tag{1.1}$$

Based in the current density we can calculate the electric current in the 'a' cross section

$$I = \int_A \boldsymbol{J} \cdot d\boldsymbol{a} = \frac{1}{\rho} \int_A \boldsymbol{E} \cdot d\boldsymbol{a} \tag{1.2}$$

The potential difference (voltage) between the end terminals of the rod

$$V = \int_0^l \boldsymbol{E} \cdot d\boldsymbol{r} = \boldsymbol{E} \cdot \boldsymbol{l} \rightarrow E = \frac{V}{l} \tag{1.3}$$

Notice that the electric field is homogeneous within the rod. Substituting (1.3) and (1.2) to (1.1) we obtain the integral form of the Ohm's law.

$$I = \frac{1}{\rho} \cdot \frac{V}{l} \cdot a = \frac{V}{\rho \cdot {}^{l}/_{a}} = \frac{V}{R} \tag{1.4}$$

The cross section of the rod is

$$a = \frac{d^2 \cdot \pi}{4} = \frac{25 \cdot \pi}{4} = 19.625\ mm^2 \tag{1.5}$$

And the R resistance of the rod is

$$R = \rho \cdot \frac{l}{a} = 0.025 \cdot \frac{0.1}{19.625} = 1.2739 \cdot 10^{-4} = 127.39\ \mu\Omega \tag{1.6}$$

So, the electric current which flows through the aluminium rod is

$$I = \frac{V}{R} = \frac{10^{-3}}{1.2739 \cdot 10^{-4}} = 7.85\ \mathrm{A} \tag{1.7}$$

1.2 Resistance calculation in irregular shapes

Find I_{AB} then I_{CD} when 1mV is connected to AB then CD terminals. ρ_{Al} = 0.025 Ωmm²/m Surfaces, where the current enters and leaves the aluminium body, can be supposed to be equipotential.

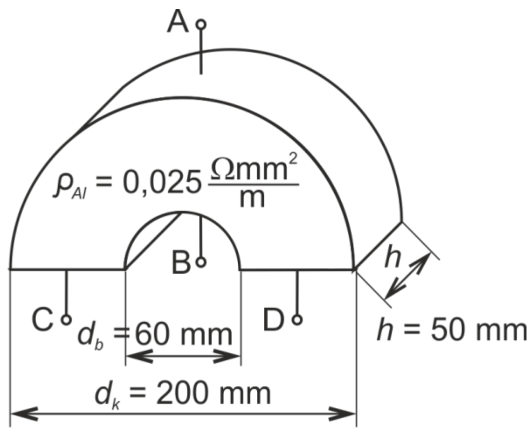

Figure 1.2 Current carrying aluminium body

Solution

Based on Ohm's law we can calculate the electric current. The first idea is to use the (1.8) integral formula of the resistance calculation.

$$R = \rho \cdot \frac{l}{A} = \frac{l}{\sigma \cdot A} \tag{1.8}$$

The integral formula can be used only with the following conditions

- Homogeneous material σ or ρ is constant
- Constant length beside cross-section
- Constant cross-section beside length

So (1.7) provides an inaccurate (mean) result for both I_{AB} and I_{CD}.

In I_{AB} calculation, the current density varies along the current path as shown in Fig. 1.3. The cross section is not constant with the current path, so we can use the mean cross section only in our (inaccurate) calculation using (1.7) in this case.

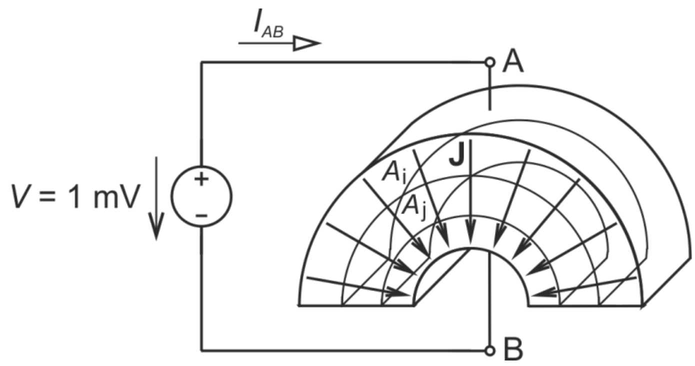

Figure 1.3 Calculating I_{AB} current

In I_{CD} calculation, the current path varies along the cross section as shown in Fig. 1.4.

so, we can use the mean cross section only in our (inaccurate!) calculation using (1.8) in this case.

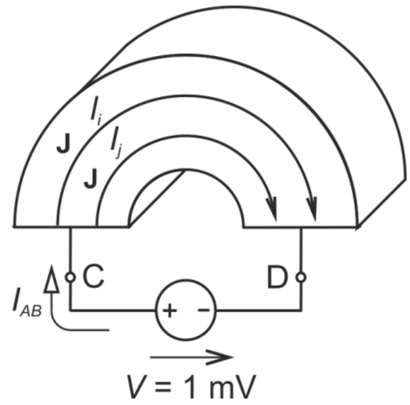

Figure 1.4 Calculating I_{CD} current

Approximate calculation of I_{AB}

$$A_m = h \cdot \frac{d_m \cdot \pi}{2} = h \cdot \frac{\frac{d_k + d_b}{2} \cdot \pi}{2} = \cdots = 10210\ mm^2 \tag{1.9}$$

$$l_{AB} = \frac{d_k - d_b}{2} = \cdots = 70\ mm = 0.07\ m \tag{1.10}$$

$$R_{ABm} = \rho \cdot \frac{l_{AB}}{A_m} = \cdots = 1.714 \cdot 10^{-7}\ \Omega \tag{1.11}$$

$$I_{ABm} = \frac{V}{R_{ABm}} = \cdots = 5.834\ kA \tag{1.12}$$

Exact calculation of I_{AB}

We can interpret Fig. 1.3 in this case as it is shown in Fig. 1.5.

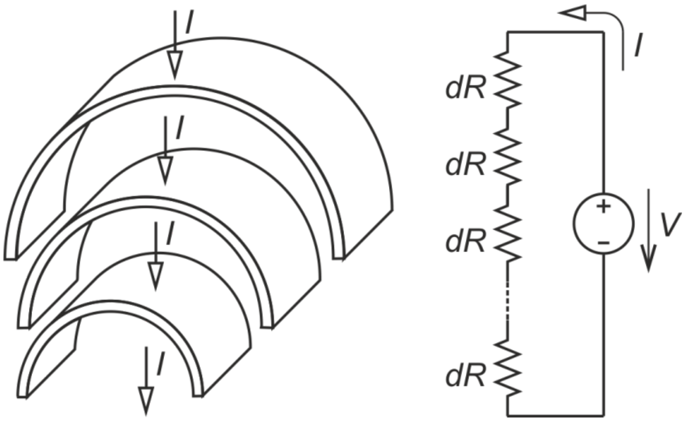

Figure 1.5 Interpretation of Fig. 1.3

For a very thin elementary layer the exact value of the resistance can be calculated according to Fig. 1.6.

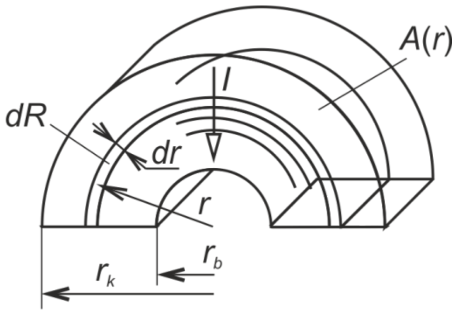

Figure 1.6 Calculating elementary resistance

$$dR_{AB} = \rho \frac{l}{A(r)} = \rho \frac{dr}{h \cdot r \cdot \pi} \tag{1.13}$$

Because of the series connected elementary resistances

$$R_{AB} = \int_{r_b}^{r_k} dR_{AB} = \int_{r_b}^{r_k} \rho \frac{dr}{h \cdot r \cdot \pi} = \frac{\rho}{h \cdot \pi} \int_{r_b}^{r_k} \frac{1}{r} dr \tag{1.14}$$

Evaluating (1.14) and substituting parameters

$$R_{AB} = \frac{\rho}{h \cdot \pi} \int_{r_b}^{r_k} \frac{1}{r} dr = \frac{\rho}{h \cdot \pi} |\ln r|_{r_b}^{r_k} = \frac{\rho}{h \cdot \pi} (\ln r_k - \ln r_b) = \frac{\rho}{h \cdot \pi} \ln \frac{r_k}{r_b} \tag{1.15}$$

$$R_{AB} = \frac{2.5 \cdot 10^{-8}}{0.05 \cdot \pi} \ln \frac{200}{60} = 1.9162 \cdot 10^{-7} \Omega \tag{1.16}$$

Finally, the exact value of the electric current is

$$I_{AB} = \frac{V}{R_{AB}} = \cdots = 5.219\ A \leftarrow (\neq 5.834\ A) \tag{1. 17}$$

Approximate calculation of I_{CD}

According to Fig. 1.4

$$l_m = \frac{d_m \cdot \pi}{2} = \frac{\frac{d_k + d_b}{2} \cdot \pi}{2} = \cdots = 204.2\ mm = 0.2042\ m \tag{1.18}$$

$$A_{CD} = h \frac{d_k - d_b}{2} = \cdots = 3500\ mm^2 \tag{1.19}$$

$$R_{CDm} = \rho \frac{l_m}{A_{CD}} = \cdots = 1.46 \cdot 10^{-6}\ \Omega \tag{1.20}$$

$$I_{CDm} = \frac{V}{R_{CDm}} = \cdots = 685\ A \tag{1.21}$$

<u>Exact calculation of I_{CD}</u>

We can interpret Fig. 1.4, in this case as is shown in Fig. 1.7.

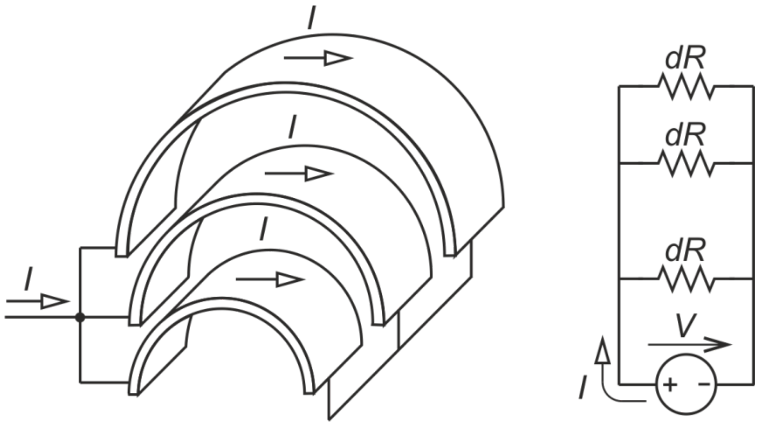

Figure 1.7 Interpretation of Fig. 1.4

For a very thin elementary layer the exact value of the conductance can be calculated according to Fig. 1.8.

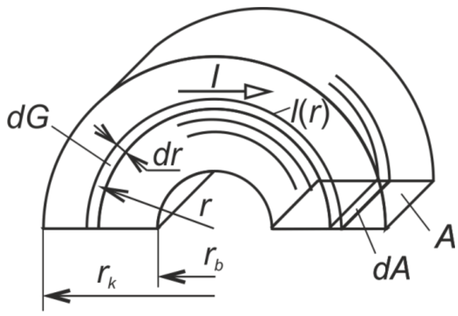

Figure 1.8 Calculating elementary conductance

$$dG_{CD} = \frac{1}{dR_{CD}} = \frac{dA}{\rho \cdot l(r)} = \frac{h \cdot dr}{\rho \cdot r \cdot \pi} \tag{1.22}$$

So, the total conductance of the body is

$$G_{CD}=\int_{rb}^{r_k} dG_{CD}=\int_{r_b}^{r_k}\frac{h\cdot dr}{\rho\cdot r\cdot\pi}=\frac{h}{\rho\cdot\pi}\int_{rb}^{rk}\frac{1}{r}dr \tag{1.23}$$

Evaluating (1.23) and substituting parameters

$$G_{CD}=\frac{h}{\rho\cdot\pi}\int_{rb}^{rk}\frac{1}{r}dr=\frac{h}{\rho\cdot\pi}[\ln r]_{r_b}^{r_k}=\frac{h}{\rho\cdot\pi}(\ln r_k-\ln r_b)=\frac{h}{\rho\cdot\pi}\ln\frac{r_k}{r_b} \tag{1.24}$$

$$G_{CD}=\frac{1.15}{2.5\cdot10^{-8}\cdot\pi}\ln\frac{200}{60}=7.665\cdot10^{5}\,S \tag{1.25}$$

$$R_{CD}=\frac{1}{G_{CD}}=1.3046\cdot10^{-6}\,\Omega \tag{1.26}$$

Finally, the exact value of the electric current is

$$I_{CD}=\frac{V}{R_{CD}}=\frac{10^{-3}}{1.3046\cdot10^{-6}}=766\,A\leftarrow(\neq 685\,A) \tag{1.27}$$

1.3 Earth resistance calculation

Calculate the earth resistance in Fig. 1.9 when the ground is wet and when it is dry.

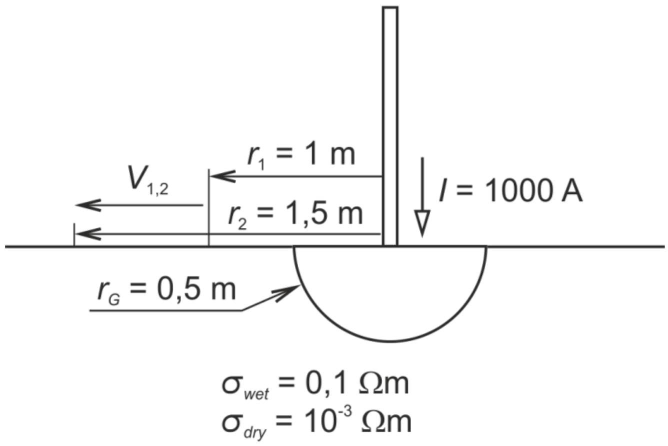

Figure 1.9 Grounded rod for earth resistance calculation

Solution

For a very thin elementary layer the exact value of the resistance can be calculated according to Fig. 1.10.

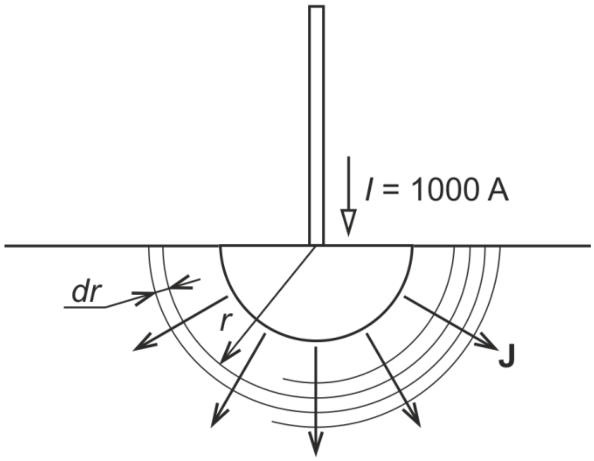

Figure 1.10 Calculating the resistance of a thin layer

$$dR = \frac{l}{\sigma A(r)} = \frac{dr}{\sigma \cdot 2\pi \cdot r^2} \tag{1.28}$$

Therefore, the total resistance of the grounded rod can be calculated according to the Fig. 1.11.

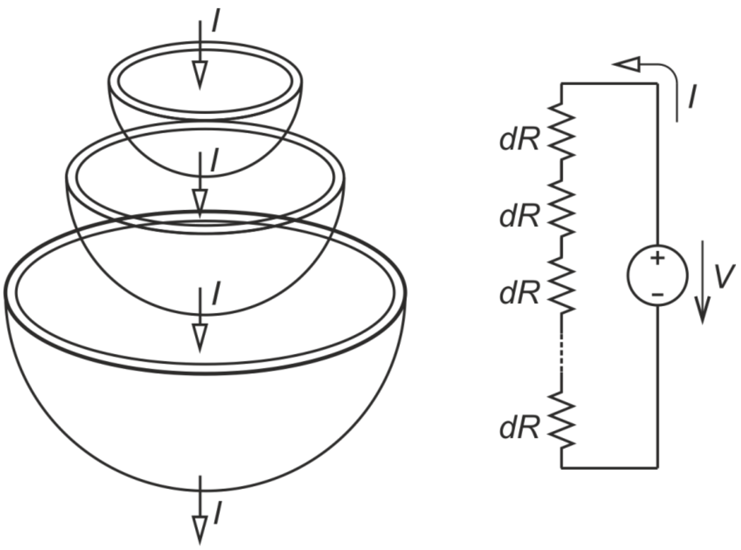

Figure 1.11 Interpretation of Fig. 1.9

$$R_f = \int_{r_G}^{\infty} \frac{dr}{\sigma \cdot 2\pi \cdot r^2} = \frac{1}{\sigma \cdot 2\pi} \int_{r_G}^{\infty} \frac{1}{r^2} dr \tag{1.29}$$

Evaluating (1.29) we find

$$R_f = \frac{1}{\sigma \cdot 2\pi} \left[-\frac{1}{r} \right]_{r_G}^{\infty} = \frac{1}{\sigma \cdot 2\pi} \left(\frac{-1}{\infty} - \frac{-1}{r_G} \right) = \frac{1}{2\pi \cdot \sigma \cdot r_G} \tag{1.30}$$

Finally, substituting parameters

$$R_{fwet} = \frac{1}{2\pi \cdot \sigma_{wet} \cdot r_G} = 3.183\ \Omega \tag{1.31}$$

$$R_{fdry} = \frac{1}{2\pi \cdot \sigma_{dry} \cdot r_G} = 318.3\ \Omega \tag{1.32}$$

1.4 Electric potential of a grounded rod

Find the electric potential of the rod in Fig. 1.9 when the ground is wet and when it is dry.

Solution

Substituting earth resistances from (1.31) and (1.32) into Ohm's law

$$V_{fwet} = I \cdot R_{fwet} = 1000 \cdot 3.183 = 3.183\ kV \tag{1.33}$$

$$V_{fdry} = I \cdot R_{fdry} = 1000 \cdot 318.3 = 318.3\ kV \tag{1.34}$$

We can see the potential of the rod is higher when the earth is dry but the potential in case of wet ground is also dangerous in the case of a 1000 A electric current, flowing through the rod.

1.5 Step voltage

Calculate the step voltage between the 1m and 1.5 m points of the lightning rod shown in Fig. 1.9 when the ground is wet and when it is dry.

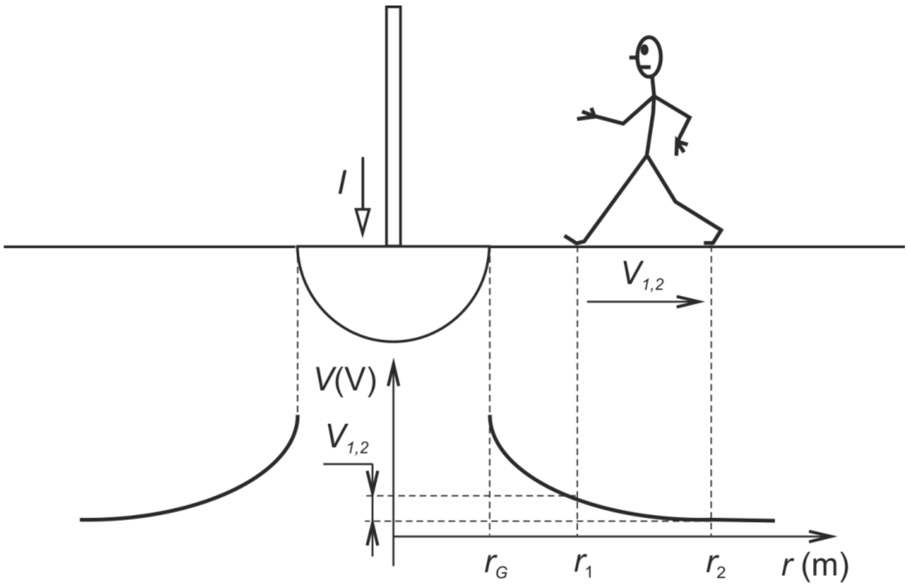

Figure 1.12 Step voltage

Solution 1

The earth resistance between r_1 and r_2 points can be calculated according to (1.30)

$$R_{1,2} = \frac{1}{2\pi \cdot \sigma} \int_{r_1}^{r_2} \frac{1}{r^2} dr = \frac{1}{2\pi \cdot \sigma} \left[-\frac{1}{r} \right]_{r_1}^{r_2} = \frac{1}{2\pi \cdot \sigma} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tag{1.35}$$

Substituting parameters into (1.35) of wet and dry ground conditions

$$R_{1,2wet} = \frac{1}{2\pi \cdot \sigma_{wet}}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{1}{2\pi \cdot 0.1}\left(\frac{1}{1} - \frac{1}{1.5}\right) = 0.531\ \Omega \tag{1.36}$$

$$R_{1,2dry} = \frac{1}{2\pi \cdot \sigma_{dry}}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{1}{2\pi \cdot 10^{-3}}\left(\frac{1}{1} - \frac{1}{1.5}\right) = 53.1\ \Omega \tag{1.37}$$

So, the step voltages in case of wet and dry ground

$$V_{1,2wet} = I \cdot R_{1,2wet} = 1000 \cdot 0.531 = 531\ V \tag{1.38}$$

$$V_{1,2dry} = I \cdot R_{1,2dry} = 1000 \cdot 53.1 = 53.1\ kV \tag{1.39}$$

The step voltage can also be very dangerous with 1000 A flowing through the rod.

Solution 2

The electric current density at distance r from the surface can be calculated according to (1.40)

$$J(r) = \frac{I}{A(r)} = \frac{I}{2\pi \cdot r^2} \tag{1.40}$$

The electric field according to Ohm's law is

$$E(r) = \frac{1}{\sigma} J(r) = \frac{I}{\sigma \cdot 2\pi \cdot r^2} \tag{1.41}$$

And the potential difference i.e. the step voltage between r_1 and r_2 is

$$V_{1,2} = \int_{r_1}^{r_2} E(r)dr = \int_{r_1}^{r_2} \frac{I}{\sigma \cdot 2\pi \cdot r^2} dr = \frac{I}{2\pi \cdot \sigma} \int_{r_1}^{r_2} \frac{1}{r^2} dr = \frac{I}{2\pi \cdot \sigma}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{1.42}$$

Obviously (1.42) gives the same result for the step voltages as we already calculated in (1.38) and (1.39).

1.6 Dissipated power of a lightning rod

Calculate the dissipated power through the earth for Fig. 1.9 when the ground is wet and when it is dry.

Solution 1

We already calculated the electric potential of the rod in wet and dry conditions in (1.33) and (1.34). The dissipated power, accordingly, are the following.

$$P_{wet} = V_{fwet} \cdot I = 3183 \cdot 1000 = 3.183\ MW \tag{1.43}$$

$$P_{dry} = V_{fdry} \cdot I = 318300 \cdot 1000 = 318.3\ MW \tag{1.44}$$

Solution 2

The differential form of Joule's law gives the value of electric power density at a certain point of the electric field as shown in (1.45).

$$p(r) = E(r) \cdot J(r) = \frac{I}{\sigma \cdot 2\pi \cdot r^2} \cdot \frac{I}{2\pi \cdot r^2} = \frac{I^2}{\sigma \cdot 4\pi^2 r^4} \tag{1.45}$$

The total power, in volume, is given by (1.46)

$$P_V = \int_V p(r)\, dV \ , \qquad dV = A \cdot dr = 2\pi \cdot r^2 \cdot dr \tag{1.46}$$

Substituting (1.45) into (1.46) and calculating the equation

$$P_V = \int_{r_G}^{\infty} \frac{I^2}{\sigma \cdot 4\pi^2 r^4} \cdot 2\pi \cdot r^2 \cdot dr = \frac{I^2}{2\pi \cdot \sigma} \int_{r_G}^{\infty} \frac{1}{r^2} dr = \frac{I^2}{2\pi \cdot \sigma \cdot r_G} \tag{1.47}$$

$$P_{wet} = \frac{1000^2}{0.1 \cdot 2\pi \cdot 0.5} = 3.183\ MW \tag{1.48}$$

$$P_{dry} = \frac{1000^2}{10^{-3} \cdot 2\pi \cdot 0.5} = 318.3\ MW \tag{1.49}$$

Obviously, (1.48) and (1.49) provide the same results as we obtained from (1.43) and (1.44).

2. Basic Laws of Electric Circuits

2.1 Circuit analysis using Kirchhoff's method 1

Find voltages v_1 and v_2 in the single loop electrical circuit, shown in Fig. 2.1.

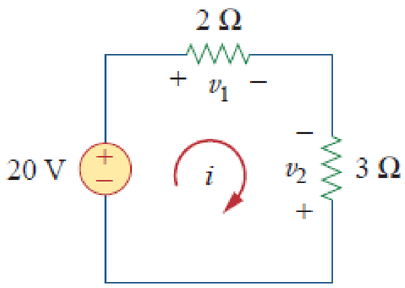

Figure 2.1 Electrical circuit with a single loop

Solution

The characteristic equations of the resistors are the following.

$$v_1 = 2 \cdot i\,, \qquad v_2 = -3 \cdot i \tag{2.1}$$

Applying KVL for the single loop we have (2.2)

$$-20 + v_1 - v_2 = 0 \tag{2.2}$$

Substituting (2.1) into (2.2)

$$-20 + 2 \cdot i - (-3 \cdot i) = 0 \tag{2.3}$$

Thus,

$$i = \frac{20}{5} = 4\,A \rightarrow v_1 = 2 \cdot 4 = 8\,V\,, v_2 = -3 \cdot 4 = -12\,V \tag{2.4}$$

2.2 Circuit analysis using Kirchhoff's method 2

Find the currents and voltages in the circuit shown in Fig. 2.2.

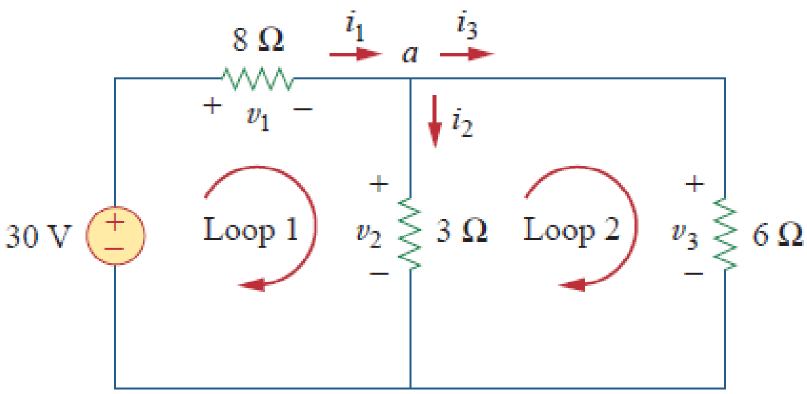

Figure 2.2 Circuit with two independent loops

Solution

The circuit have two independent loops and one independent node. Applying KVL for Loop 1

$$-30 + v_1 + v_2 = 0 \tag{2.5}$$

Applying KVL for Loop 2

$$-v_2 + v_3 = 0 \rightarrow v_2 = v_3 \tag{2.6}$$

Applying KCL for Node ‘a’

$$i_1 - i_2 - i_3 = 0 \tag{2.7}$$

The characteristic equations of the resistors

$$v_1 = 8 \cdot i_1\,, \qquad v_2 = 3 \cdot i_2\,, \qquad v_3 = 6 \cdot i_3 \tag{2.8}$$

Substituting (2.8) into (2.5)

$$-30 + 8 \cdot i_1 + 3 \cdot i_2 = 0 \rightarrow i_1 = \frac{30 - 3 \cdot i_2}{8} \tag{2.9}$$

Substituting (2.8) into (2.6)

$$6 \cdot i_3 = 3 \cdot i_2 \rightarrow i_3 = 0.5 \cdot i_2 \tag{2.10}$$

And finally, substituting (2.9) and (2.10) into (2.7)

$$\frac{30 - 3 \cdot i_2}{8} - i_2 - 0.5 \cdot i_2 = 0 \tag{2.10}$$

Thus,

$$i_2 = 2\,A\,, \qquad i_3 = 1\,A\,, \qquad v_1 = 24\,V, \qquad v_2 = v_3 = 6\,V \tag{2.11}$$

2.3 Circuit analysis using Kirchhoff’s method 3

Calculate the currents and voltages in the circuit shown in Fig. 2.3.

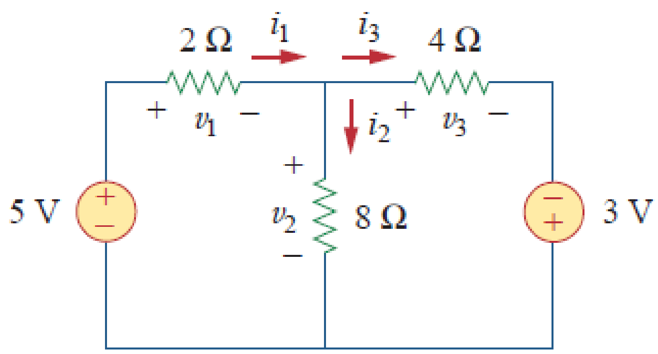

Figure 2.3 Electrical circuit with two sources

Solution

Applying KVL for independent loops and KCL for a single independent node

$$-5 + v_1 + v_2 = 0 \tag{2.12}$$

$$-3 - v_2 + v_3 = 0 \tag{2.13}$$

$$i_1 = i_2 + i_3 \tag{2.14}$$

The resistors' characteristics are the following

$$v_1 = 2i_1\,, \qquad v_2 = 8i_2, \qquad v_3 = 4i_3 \tag{2.15}$$

Substituting (2.15) into (2.12) and (2.13)

$$2i_1 + 8i_2 = 5 \tag{2.16}$$

$$-8i_2 + 4i_3 = 3 \rightarrow i_3 = \frac{3 + 8i_2}{4} \tag{2.17}$$

From (2.16) and (2.14) we obtain

$$2(i_2 + i_3) + 8i_2 = 5 \rightarrow 10i_2 + 2i_3 = 5 \tag{2.18}$$

From (2.18) and (2.17) we can write

$$10i_2 + \frac{3 + 8i_2}{2} = 5 \tag{2.19}$$

Thus,

$$20i_2 + 3 + 8i_2 = 10 \rightarrow i_2 = 0.25\ A \tag{2.20}$$

$$i_3 = 1.25\ A\,, \qquad i_1 = 1.5\ A \tag{2.21}$$

$$v_1 = 3\ V\,, \qquad v_2 = 2\ V\,, \qquad v_3 = 5\ V \tag{2.22}$$

2.4 Equivalent resistance

Find the equivalent resistance for the circuit shown in Figure 2.4.

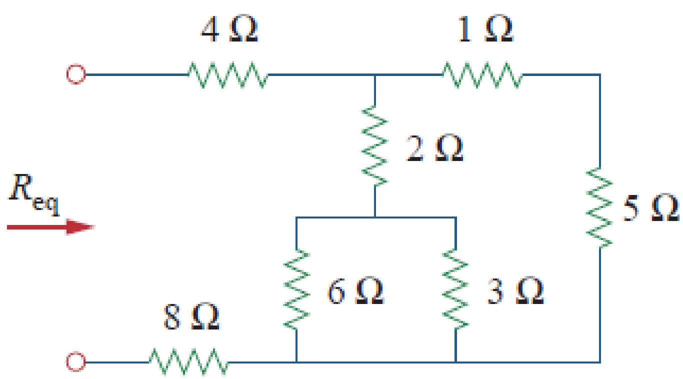

Figure 2.4 Equivalent resistance calculation

Solution

We can calculate the parallel equivalent resistance of the $6-\Omega$ and $3-\Omega$ resistances

$$\frac{3 \cdot 6}{3+6} = 2\ \Omega$$

The circuit is shown in Fig. 2.5.

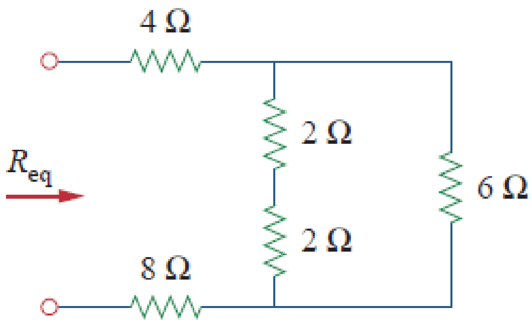

Figure 2.5 Circuit with the sub-equivalent 1

The equivalent resistance of the series connected 2-Ω resistors is $2+2=4\ \Omega$ that is connected in parallel to the 6-Ω resistor, so

$$\frac{4 \cdot 6}{4+6} = 2.4\ \Omega$$

as shown in Fig. 2.6.

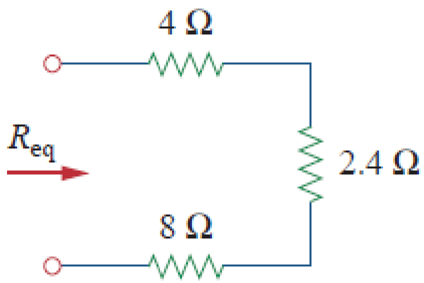

Figure 2.6 Circuit with the sub-equivalent 2

So, the equivalent resistance of the series connected resistors is

$$R_{eq} = 4 + 2.4 + 8 = 14.4\ \Omega$$

2.5 Electric power calculation

Find i_O and v_O in the circuit shown in Fig. 2.7. Calculate the power dissipated in the 3-Ω-resistor.

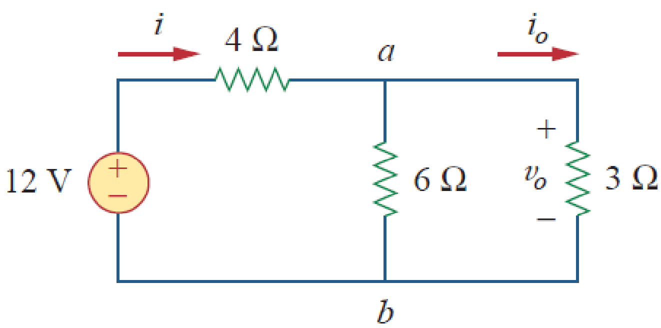

Figure 2.7 Circuit for calculation Example 2.5

Solution

We can substitute the parallel resistors with a single equivalent resistor as shown in Fig. 2.8.

$$6 \times 3 = \frac{6 \cdot 3}{6+3} = 2\,\Omega$$

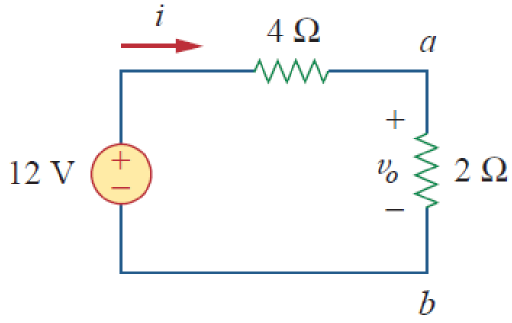

Figure 2.8 The equivalent circuit

According to the voltage division we can calculate v_0 as the following

$$v_0 = 12\,V \cdot \frac{2\,\Omega}{(2+4)\,\Omega} = 4\,V$$

and applying Ohm's law for the 3-Ω resistor

$$i_0 = \frac{v_0}{3} = 1.33\,A$$

Alternatively, we can also use current division

$$i_0 = i \cdot \frac{6\,\Omega}{(6+3)\,\Omega}$$

For this we need the total current in the circuit, that is

$$i = \frac{12\,V}{(4+2)\,\Omega} = 2\,A$$

thus,

$$i_0 = 2\,A \cdot \frac{6\,\Omega}{(6+3)\Omega} = 1.33\,A$$

that gives the same value as we calculated previously.

Finally, the electric power at the 3-Ω resistor is

$$p_0 = v_0 \cdot i_0 = 4 \cdot 1.33 = 5.33\,W$$

2.6 Wye-delta equivalent transformation

Obtain current *i* in the circuit of Fig. 2.9.

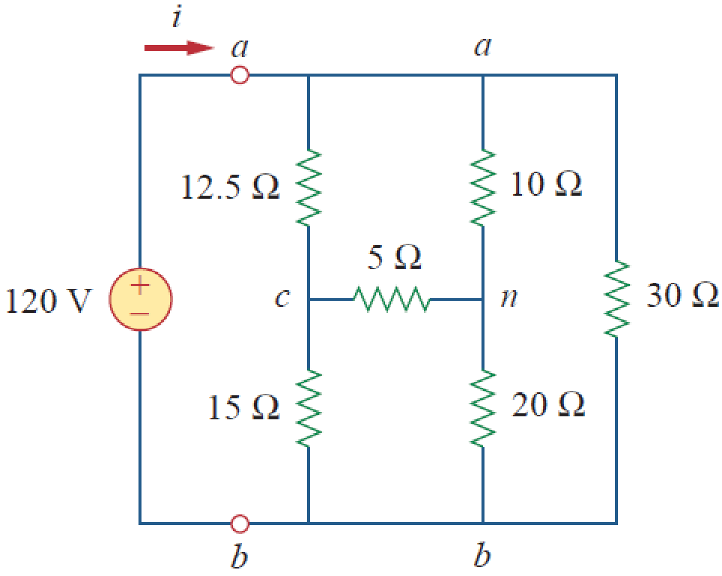

Figure 2.9 Circuit with triple-degree nodes

Solution

For current calculation, we need to determine the equivalent resistance of the circuit. But, because the circuit contains triple-degree nodes, neither series nor parallel sub-equivalents can be found. We have to change the structure of the circuit with equivalent wye-delta (or delta-wye) transformation. Let's look at the (5, 10, 20)-Ω wye sub-circuit to transform it into its delta equivalent.

$$G_\Delta = \frac{1}{5} + \frac{1}{10} + \frac{1}{20} = 0.35\ S$$

$$G_{ac} = \frac{\frac{1}{5} \cdot \frac{1}{10}}{G_\Delta} = \frac{0.02}{0.35} \rightarrow R_{ac} = \frac{0.35}{0.02} = 17.5\ \Omega$$

$$G_{bc} = \frac{\frac{1}{5} \cdot \frac{1}{20}}{G_\Delta} = \frac{0.01}{0.35} \rightarrow R_{ac} = \frac{0.35}{0.01} = 35\ \Omega$$

$$G_{ab} = \frac{\frac{1}{10} \cdot \frac{1}{20}}{G_\Delta} = \frac{0.005}{0.35} \rightarrow R_{ac} = \frac{0.35}{0.005} = 70\ \Omega$$

The transformed circuit is shown in Fig. 2.10.

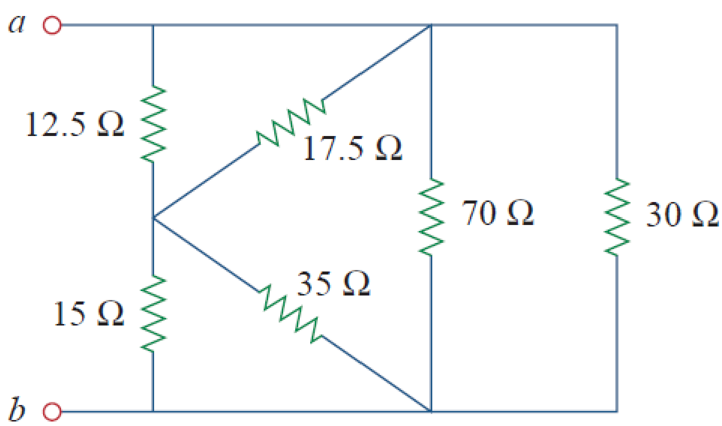

Figure 2.10 Transformed equivalent circuit

Now, we can find parallel connected elements in Fig. 2.10

$$70 \times 30 = \frac{70 \cdot 30}{100} = 21\ \Omega$$

$$12.5 \times 17.5 = \frac{12.5 \cdot 17.5}{30} = 7.292\ \Omega$$

$$15 \times 35 = \frac{15 \cdot 35}{50} = 10.5\ \Omega$$

This results in the following, in Fig. 2.11

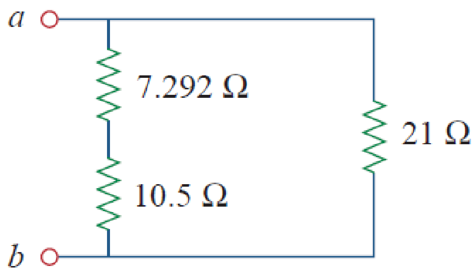

Figure 2.11 Result of parallel equivalents

Calculating the total equivalent resistance in Fig. 2.11 gives the following

$$R_{ab} = (7.292 + 10.5) \times 21 = \cdots = 9.632\ \Omega$$

And finally, the electric current is

$$i = \frac{120\ V}{R_{ab}} = 12.458\ A$$

3. Methods and Theorems in Circuit Analysis

3.1 Nodal analysis

Using nodal analysis, calculate the node voltages and the branch currents in the circuit shown in Fig. 3.1.

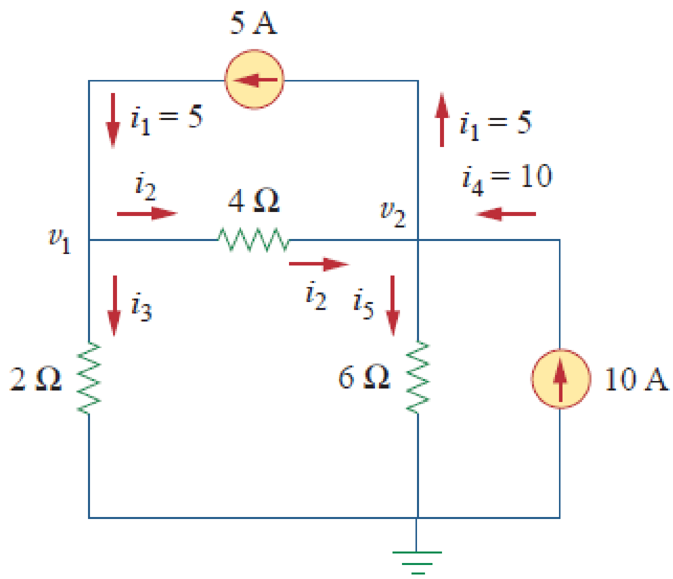

Figure 3.1 Example for nodal analysis

Solution

First of all, we must decide the direction of the current, as is shown in Fig, 3.1. If any of these directions is supposed, in contrast to 'real life' then its value will result in a negative in the calculation. Note that with the supposed direction of i_2 we also used a v_1 potential higher than v_2. By applying KCL for node 1, having potential v_1 we can write (3.1).

$$i_1 = i_2 + i_3 \tag{3.1}$$

Substituting circuit parameters into (3.1)

$$5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2} \tag{3.2}$$

that can be written as

$$20 = v_1 - v_2 + 2v_1 \tag{3.3}$$

thus

$$3v_1 - v_2 = 20 \tag{3.4}$$

Applying KCL for node 2, having potential v_2 we can write (3.5).

$$i_2 + i_4 = i_1 + i_5 \tag{3.5}$$

Substituting circuit parameters into (3.5)

$$\frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6} \tag{3.6}$$

that can be written as

$$3v_1 - 3v_2 + 120 = 60 + 2v_2 \tag{3.7}$$

thus

$$-3v_1 + 5v_2 = 60 \tag{3.8}$$

Our task is to solve equations of (3.4) and (3.8) to determine v_1 and v_2 node voltages.

Method 1

Adding (3.4) and (3.8) we have

$$4v_2 = 80 \rightarrow v_2 = 20\,V \tag{3.9}$$

From (3.4) we can express vis$_1$

$$3v_1 - 20 = 20 \rightarrow v_1 = 13.33\,V \tag{3.10}$$

Method 2 *(Cramer's rule)*

(3.4) and (3.8) can be written in matrix form as

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix} \tag{3.11}$$

The determinant of the coefficient matrix

$$\Delta = det \begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} = 15 - 3 = 12 \tag{3.12}$$

Substituting the first column of the coefficient matrix with the constant vector the determinant is

$$\Delta_1 = det \begin{bmatrix} 20 & -1 \\ 60 & 5 \end{bmatrix} = 100 + 60 = 160 \tag{3.13}$$

Substituting the second column of the coefficient matrix with the constant vector the determinant is

$$\Delta_2 = det \begin{bmatrix} 3 & 20 \\ -3 & 60 \end{bmatrix} = 180 + 60 = 240 \tag{3.14}$$

So the nodal potentials, according to the Cramer's rule, can be calculated as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{160}{12} = 13.33\ V \tag{3.15}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{240}{12} = 20\ V \tag{3.16}$$

The result is the same as we calculated in (3.9) and (3,10).

Finally, to determine the branch currents.

$$i_1 = 5\ A \tag{3.17}$$

$$i_2 = \frac{v_1 - v_2}{4} = -1.67\ A \tag{3.18}$$

$$i_3 = \frac{v_1}{2} = 6.67\ A \tag{3.19}$$

$$i_4 = 10\ A \tag{3.20}$$

$$i_5 = \frac{v_2}{6} = 3.33\ A \tag{3.21}$$

Note that value of i_2 is negative which means that current direction is contrary to what we have assumed.

3.2 Nodal analysis using super node method

Find the node voltages for the circuit shown in Fig 3.2.

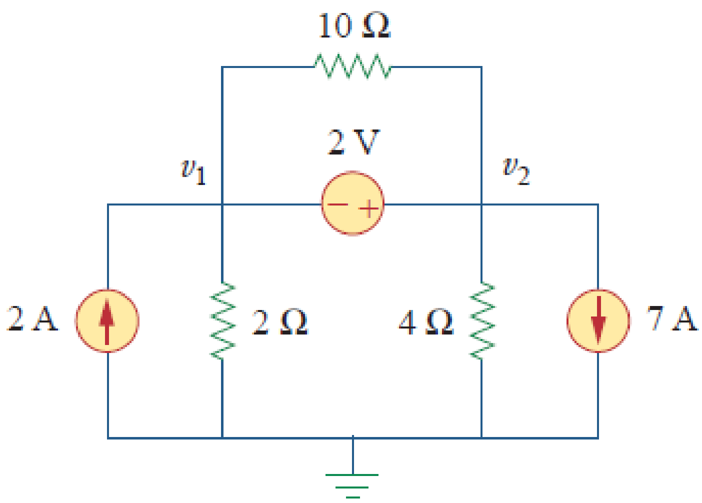

Figure 3.2 Circuit with super node

Solution

Note that a 2-V voltage source is connected directly between the node 1 (with potential v_1) and node 2 (with potential v_2) so we can substitute these nodes with a super node as shown in Fig. 3.3.

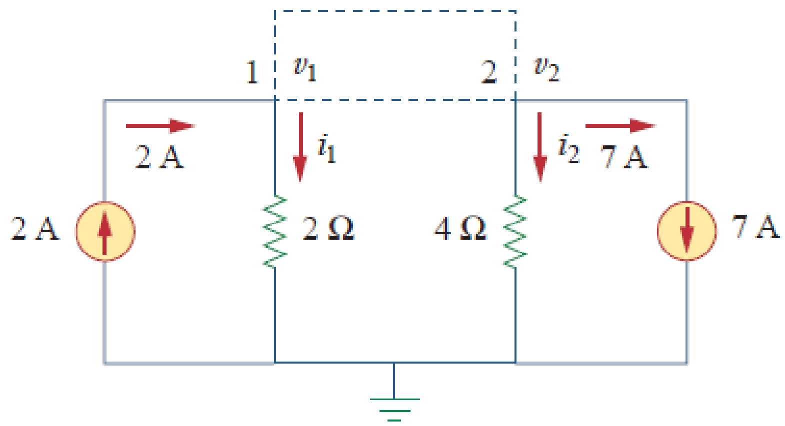

Figure 3.3 Super node

Applying KCL for the super node

$$2 = i_1 + i_2 + 7 \tag{3.22}$$

Substituting circuit parameters into (3.22)

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \rightarrow 8 = 2v_1 + v_2 + 28 \tag{3.23}$$

thus

$$v_2 = -20 - 2v_1 \tag{3.24}$$

According to the super node condition we can write that

$$v_2 = v_1 + 2 \tag{3.25}$$

From (3.24) and (3.25)

$$-20 - 2v_1 = v_1 + 2 \tag{3.26}$$

So, the node potentials can be expressed as

$$3v_1 = -22 \rightarrow v_1 = -7.33\ V\ ,\ \ v_2 = v_1 + 2 = -5.33\ V \tag{3.27}$$

3.3 Mesh analysis

Find the branch currents (I_1, I_2, I_3) using mesh analysis.

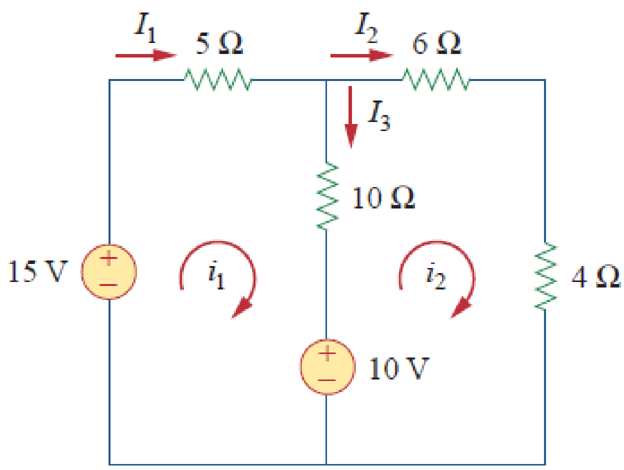

Figure 3.4 Circuit example for mesh analysis

Solution

Applying KVL for mesh 1 (current i_1)

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0 \rightarrow 3i_1 - 2i_2 = 1 \tag{3.28}$$

Applying KVL for mesh 2 (current i_2)

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0 \rightarrow -i_1 + 2i_2 = 1 \tag{3.29}$$

(3.28) and (3.28) in matrix form for using Cramer's rule in the solution

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.30}$$

The coefficient determinant

$$\Delta = det \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = 4 \tag{3.31}$$

Substituting the constant vector into the first column of the coefficient matrix

$$\Delta_1 = det \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} = 4 \tag{3.32}$$

Substituting the constant vector into the second column of the coefficient matrix

$$\Delta_2 = det \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = 4 \tag{3.33}$$

The mesh currents are the following

$$i_1 = \frac{\Delta_1}{\Delta} = 1\,A\,, \qquad i_2 = \frac{\Delta_2}{\Delta} = 1\,A \tag{3.34}$$

Finally, we must calculate the branch currents

$$I_1 = i_1 = 1\,A\,, \qquad I_2 = i_2 = 1\,A\,, \qquad I_3 = i_1 - i_2 = 0 \tag{3.35}$$

3.4 Mesh analysis using the super mesh method

Find i_1, i_2, i_3, i_4 using mesh analysis in the circuit of Fig. 3.5.

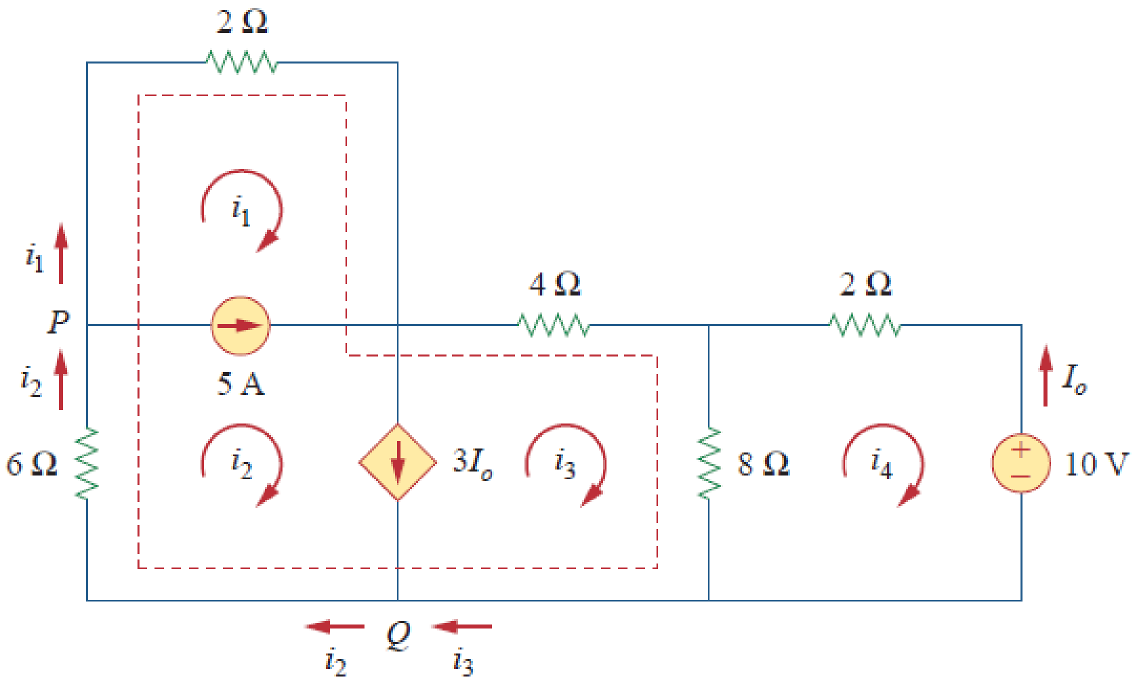

Figure 3.5 Mesh analysis with super mesh method

Solution

Because mesh 1, mesh 2 and mesh 3 build a super mesh we can write the KVL for that super mesh as follows

$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0 \tag{3.36}$$

From which we get (3.37)

$$i_1 + 3i_2 + 6i_3 - 4i_4 = 0 \tag{3.37}$$

Additional node equation for node P

$$i_2 = i_1 + 5 \tag{3.38}$$

Additional node equation for node Q

$$i_2 = i_3 + 3I_0 \tag{3.39}$$

According to the condition of the dependent generator we can state that

$$I_0 = -i_4 \rightarrow i_2 = i_3 - 3i_4 \tag{3.40}$$

The KVL for mesh 4

$$2i_4 + 8(i_4 - i_3) + 10 = 0 \tag{3.41}$$

from which we get (3.42)

$$5i_4 - 4i_3 = -5 \tag{3.42}$$

Solving (3.37) and (3.42) we can get the mesh currents.

$$i_1 = -7.5\ A\,, \qquad i_2 = -2.5\ A, i_3 = 3.93\ A\,, \qquad i_4 = 2.143\ A \tag{3.43}$$

3.5 Superposition principle

Using the superposition principle, find *i* in the circuit on Fig. 3.6.

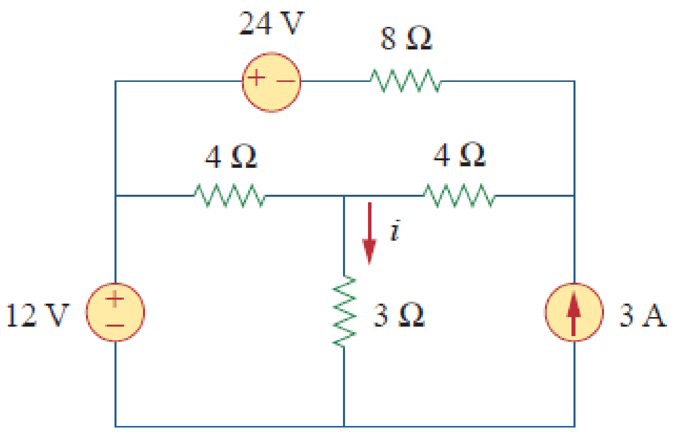

Figure 3.6 Circuit for superposition example

Solution

Using the superposition principle, we take into consideration the sources separately and the final result will be given as the addition (superposition) of partial results given by (3.44).

$$i = i_1 + i_2 + i_3\,,\ \ i_1 = ? \tag{3.44}$$

Let's start the calculation with i_1 based on a 12 V source. Other sources are substituted by their energy-free compliance, i.e. the voltage source is short circuited and current source is an open circuit as is shown in Fig. 3.7.

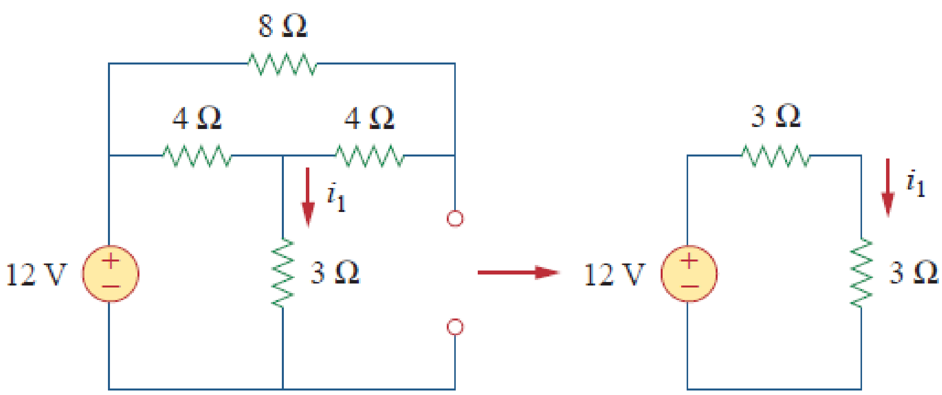

Figure 3.7 Calculation of i_1

$$i_1 = \frac{12}{6} = 2\ A \tag{3.45}$$

For calculating i_2 we can use Fig. 3.8.

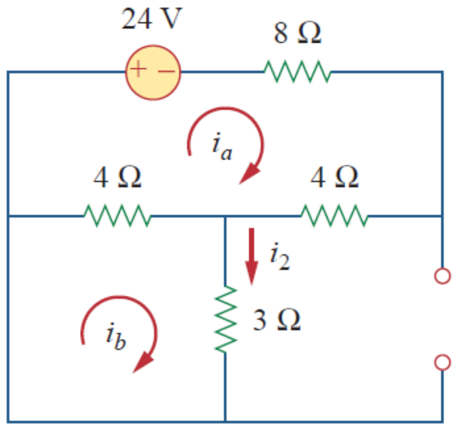

Figure 3.8 Calculating the effect of 24 V source

Applying KVL for mesh 'a' we get (3.46)

$$16i_a - 4i_b + 24 = 0 \rightarrow 4i_a - i_b = -6 \tag{3.46}$$

Applying KVL for mesh 'b' we get (3.47)

$$7i_b - 4i_a = 0 \rightarrow \mathbf{(2)}:\ i_a = \frac{7}{4}i_b \tag{3.47}$$

Substituting (3.47) into (3.36)

$$i_2 = i_b = -1 \tag{3.48}$$

For calculating i_2 we can use Fig. 3.9.

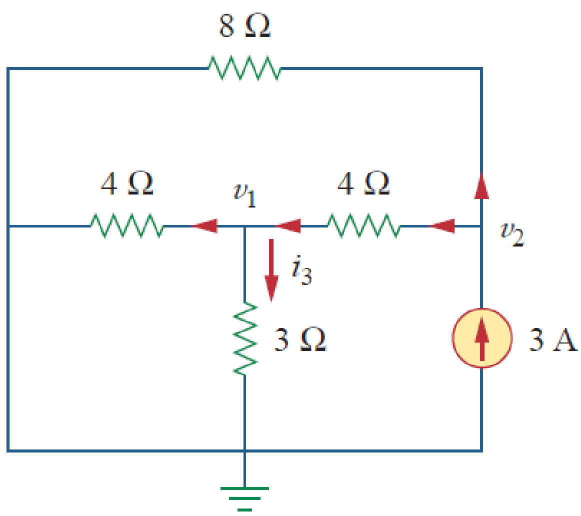

Figure 3.9 Calculating the effect of a 3 A source.

Applying KCL for node 2 (node with v_2 potential)

$$3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4} \rightarrow 24 = 3v_2 - 2v_1 \tag{3.49}$$

Applying KCL for node 1 (node with v_1 potential)

$$\frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3} \rightarrow (\mathbf{2}): v_2 = \frac{10}{3} v_1 \tag{3.50}$$

From (3.49) and (3.50)

$$v_1 = 3 \rightarrow i_3 = \frac{v_3}{3} = 1\,A \tag{3.51}$$

And finally, we can calculate current *i* based on the superposition of i_1, i_2 and i_3.

$$i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2\,A \tag{3.52}$$

3.6 Source transformation

Using source transformation to find v_x in the circuit shown in Fig. 3.10.

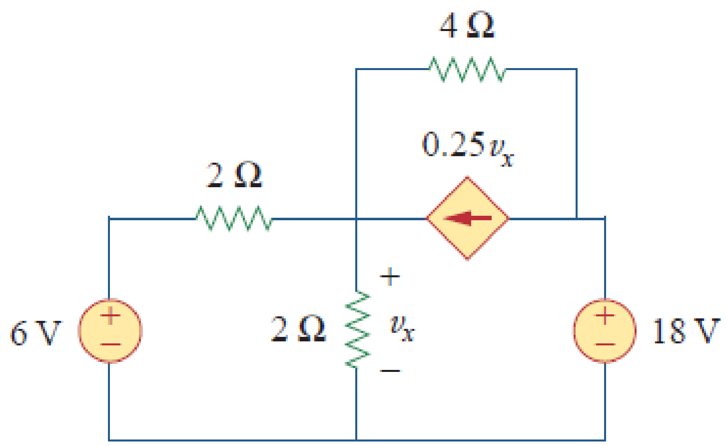

Figure 3.10 Example circuit for source transformation

Solution

The first step is to apply Thevenin-Norton transformations for the sub-circuits resulting in the changed structure shown in Fig. 3.11. The internal resistors of Thevenin and Norton equivalents are the same. The source (internal) current of the Norton generator is given by the short circuit current of the Thevenin equivalent. The source (internal) voltage of Thevenin generator is given by open circuit voltage of the Norton equivalent.

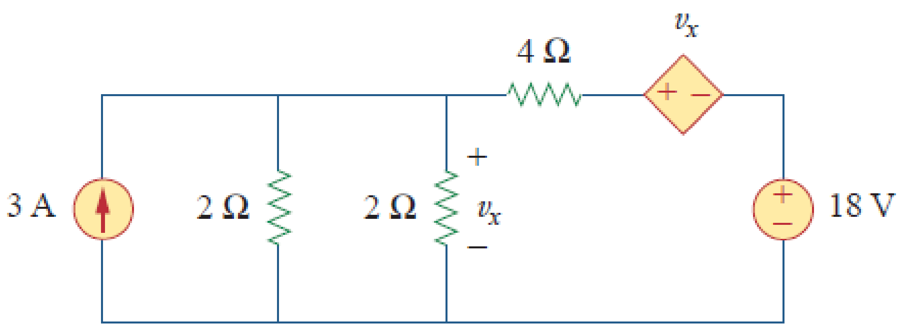

Figure 3.11 Applied Thevenin-Norton transformations on the circuit in Fig. 3.10

Upon application of Norton-Thevenin transformation for the left side Norton generator, the circuit will be as given in Fig. 3.12.

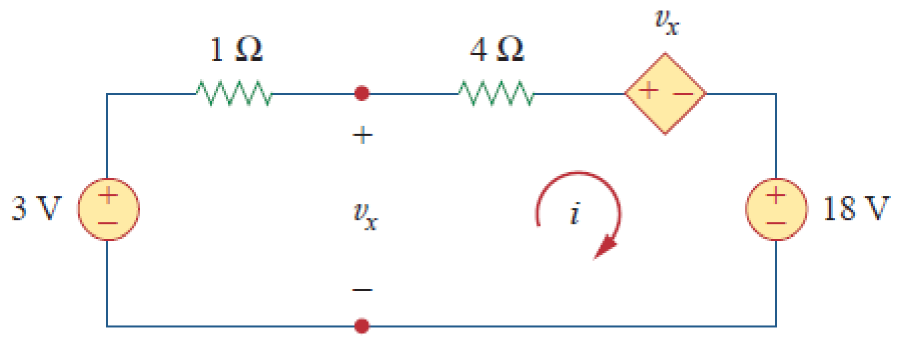

Figure 3.12 Norton-Thevenin transformation on the circuit in Fig. 3.11

Applying KVL for the circuit loop we get (3.53)

$$-3+5i+v_x+18=0 \tag{3.53}$$

We can write an additional loop equation for the left side internal loop as (3.54)

$$-3+1i+v_x=0 \tag{3.54}$$

Substituting v_x from (3.54) into (3.53) we can express the current as

$$15+5i+3-i=0 \rightarrow i=-4.5\ A \tag{3.55}$$

Another way is to write KVL for the right side internal loop from which we also can express the electric current.

$$-v_x+4i+v_x+18=0 \rightarrow i=-4.5\ A \tag{3.56}$$

The result is the same. Finally, we get the v_x voltage

$$v_x=3-i=7.5\ V \tag{3.57}$$

3.7 Thevenin's theorem

Find the Thevenin equivalent circuit at terminals *a-b* on the circuit shown in Fig. 3.13.

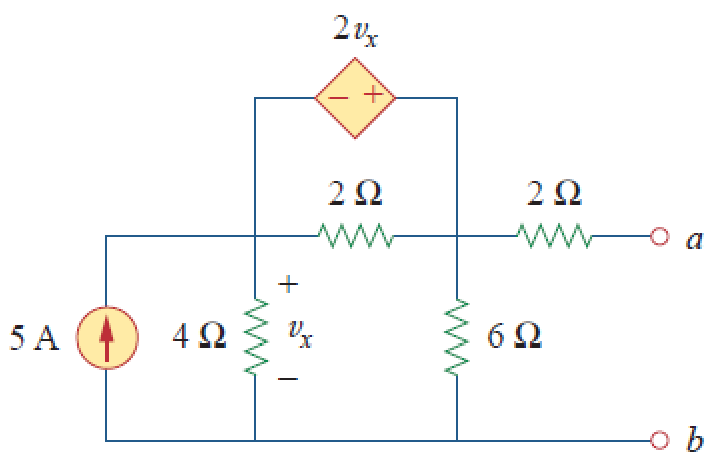

Figure 3.13 Circuit for finding the Thevenin equivalent

Solution

To find the Thevenin equivalent we have to calculate the source (Thevenin) voltage and the internal (Thevenin) resistance of Thevenin generator. Let's start with the calculation of the Thevenin resistance. For this we have to eliminate independent current sources but the dependent source will be remain in the circuit. In this case, the R_{ab} can be calculated using an external source, connected to terminals a-b as shown in Fig. 3.14.

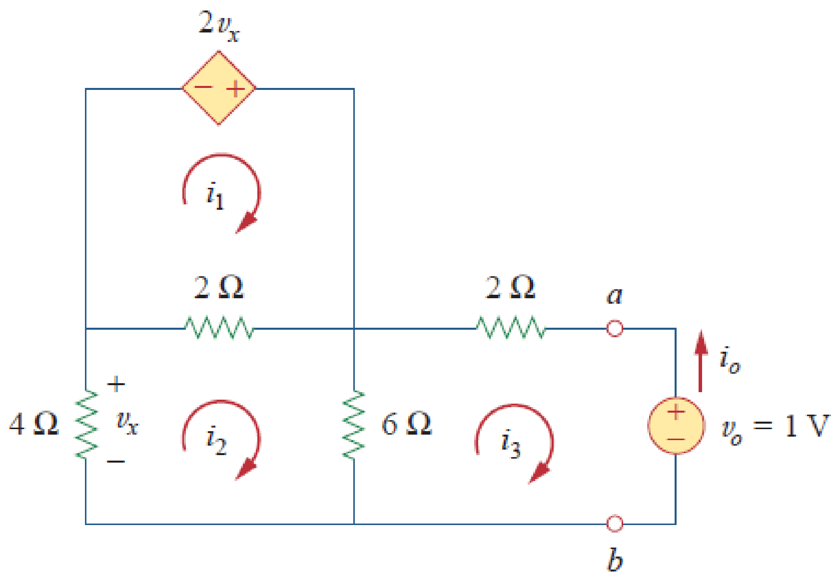

Figure 3.14 Thevenin resistance calculation

Applying KVL for mesh 1

$$-2v_X + 2(i_1 - i_2) = 0 \rightarrow v_X = i_1 - i_2 \tag{3.58}$$

But, because v_x is measured on the 4-Ω resistor

$$-4i_2 = v_X = i_1 - i_2 \rightarrow i_1 = -3i_2 \tag{3.59}$$

Applying KVL for mesh 2

$$4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0 \tag{3.60}$$

Applying KVL for mesh 3

$$6(i_3 - i_2) + 2i_3 + 1 = 0 \tag{3.61}$$

Thus,

$$i_3 = -\frac{1}{6}\,A \rightarrow i_0 = -i_3 = \frac{1}{6}\,A \tag{3.62}$$

And finally, the Thevenin resistance is

$$R_{Th} = \frac{1\,V}{i_0} = 6\,\Omega \tag{3.63}$$

Now, we must determine Thevenin voltage. For this we use the circuit shown in Fig. 3.15.

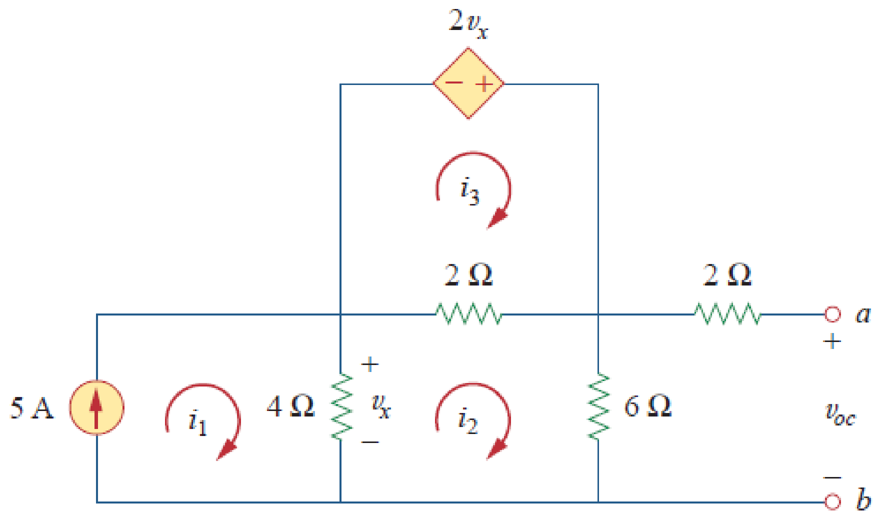

Figure 3.15 Circuit for determining Thevenin voltage

Equation for mesh 1 is simple.

$$i_1 = 5\ A \tag{3.64}$$

Equation for mesh 3 is

$$-2v_x + 2(i_3 - i_2) = 0 \rightarrow v_X = i_3 - i_2 \tag{3.65}$$

and for mesh 2 is

$$4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0 \tag{3.66}$$

that can be written as

$$12i_2 - 4i_1 - 2i_3 = 0 \tag{3.67}$$

But Ohm's law for the $4 - \Omega$ resistor is

$$4(i_1 - i_2) = v_x \tag{3.68}$$

Thus,

$$i_2 = \frac{10}{3}\ A \rightarrow V_{Th} = v_{OC} = 6i_2 = 20\ V \tag{3.69}$$

The Thevenin equivalent of the circuit in Fig. 3.13 is shown in Fig. 3.16

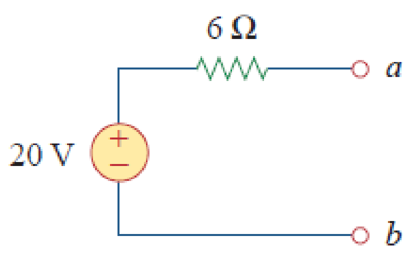

Figure 3.16 Thevenin equivalent circuit

3.8 Norton's theorem

Find the Norton equivalent of the circuit in Fig. 3.17.

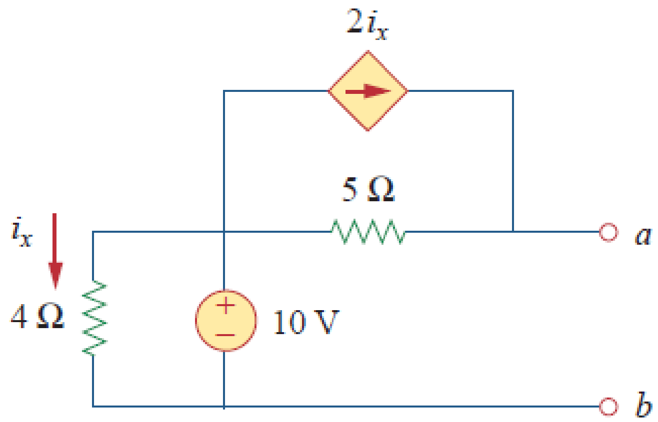

Figure 3.17 Circuit for Norton equivalent calculation

Solution

We can calculate the Norton resistance initially using the circuit shown in figure 3.18.

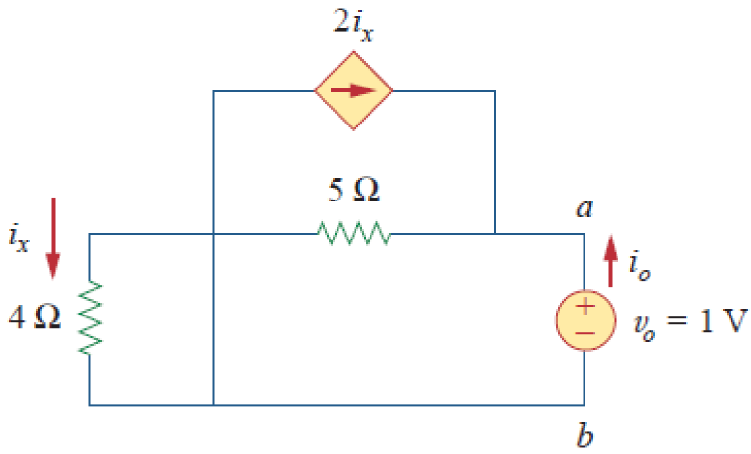

Figure 3.18 Calculating Norton resistance

As i_x is zero because of the short circuit i_0 is 0.2 A so

$$R_N = \frac{v_0}{i_0} = \frac{1}{0.2} = 5\ \Omega \tag{3.70}$$

For calculating the Norton current, we use the circuit in Fig. 3.19.

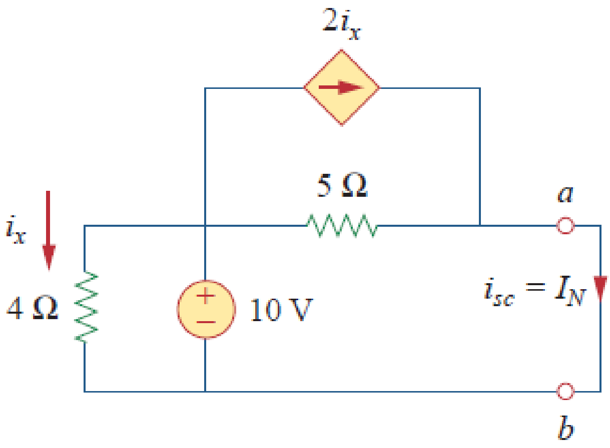

Figure 3.19 Circuit for calculating I_N

Because

$$i_x = \frac{10}{4} = 2.5\ A \tag{3.71}$$

and

$$i_{SC} = i_N = \frac{10}{5} + 2i_x = 7\ A \tag{3.72}$$

So, the Norton generator consist of 7 A current source and 5 Ω resistance.

4. Capacitors and Inductors

4.1 Stored energy in capacitors

Calculate energy stored in each capacitor of the circuit shown in Fig. 4.1.

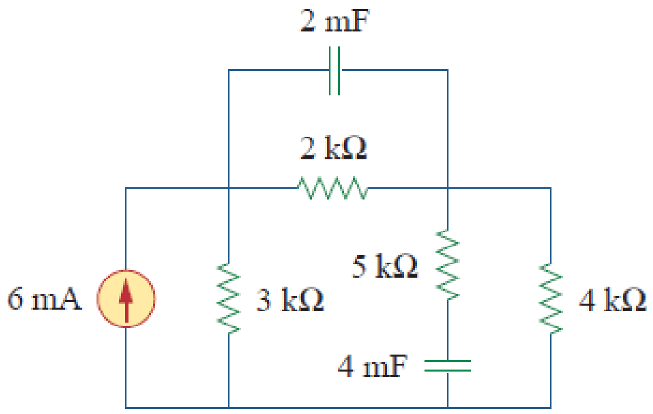

Figure 4.1 Circuit with capacitors

Solution

The DC model of the circuit is given in Fig. 4.2.

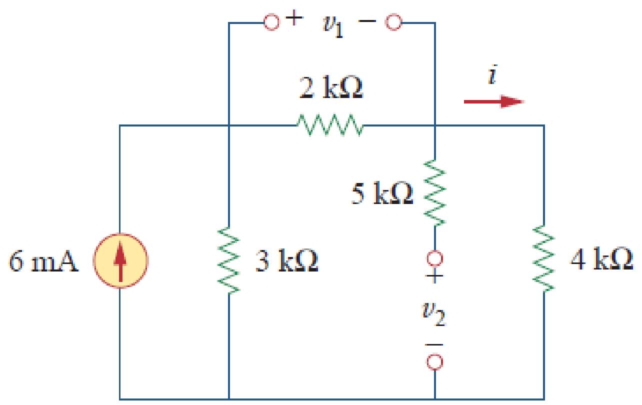

Figure 4.2 DC model of the circuit

Using current division, the *i* current can be determined as

$$i = 6\frac{3}{3+2+4} = 2\ mA \tag{4.1}$$

Thus,

$$v_1 = 2000i = 4\ V\ ,\ \ v_2 = 4000i = 8\ V \tag{4.2}$$

Finally, the energy stored in capacitors is

$$w_1 = \frac{1}{2}C_1 v_1^2 = \frac{1}{2} \cdot 2 \cdot 10^{-3} \cdot 16 = 16\ mJ \tag{4.3}$$

$$w_2 = \frac{1}{2}C_2 v_2^2 = \frac{1}{2} \cdot 4 \cdot 10^{-3} \cdot 64 = 128\ mJ \tag{4.4}$$

4.2 Equivalent capacitance

Find the equivalent capacitance between *a-b* terminals of the circuit in Fig. 4.2.

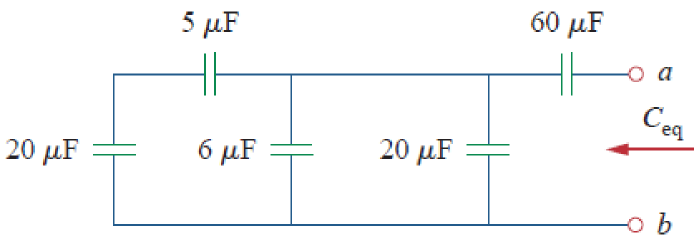

Figure 4.2 Calculating equivalent capacitance

Solution

$$\frac{20 \cdot 5}{20+5} = 4\ \mu F \rightarrow 4+6+20 = 30\ \mu F \rightarrow \frac{30 \cdot 60}{30+60} = 20\ \mu F$$

4.3 Stored energy in capacitor and inductor

Calculate the voltage across the capacitor, the current through the inductor and the energy stored in the capacitor and inductor.

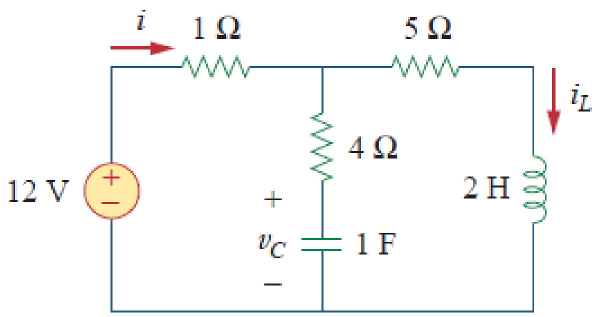

Figure 4.3 DC circuit with capacitor and inductor

Solution

The DC model of the circuit is given in Fig. 4.4.

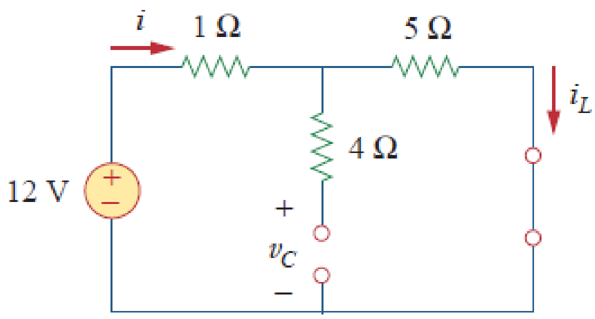

Figure 4.4 DC model of the circuit in Fig. 4.3

The electric current in the single loop is

$$i = i_L = \frac{12}{1+5} = 2\ A \tag{4.5}$$

and the voltage across the capacitor is

$$v_C = 5i = 10\,V \tag{4.6}$$

Once we have the inductors current and the capacitors voltage we can determine the stored energy in each storage element.

$$w_C = \frac{1}{2} C v_C^2 = \frac{1}{2} \cdot 1 \cdot 100 = 50\,J \tag{4.7}$$

$$w_L = \frac{1}{2} L i_L^2 = \frac{1}{2} \cdot 2 \cdot 4 = 4\,J \tag{4.8}$$

5. AC Solid State Analysis

5.1 Circuit impedance

Find the input impedance of the circuit in Fig. 5.1 when the angular frequency is 50 rad/s.

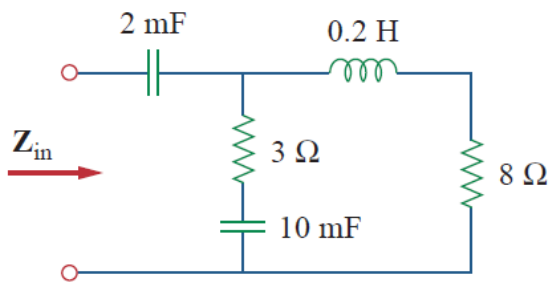

Figure 5.1 Circuit for calculating impedance

Solution

The part impedance of the 2-mF capacitor is

$$\boldsymbol{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j \cdot 50 \cdot 2 \cdot 10^{-3}} = -j10\ \Omega \tag{5.1}$$

The part impedance of the series 3-Ω resistor and 10-mF capacitor is

$$\boldsymbol{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j \cdot 50 \cdot 10 \cdot 10^{-3}} = (3 - j2)\ \Omega \tag{5.2}$$

The part impedance of the series 8-Ω resistor and 0.2-H inductor is

$$\boldsymbol{Z}_3 = 8 + j\omega L = 8 + j \cdot 50 \cdot 0.2 = (8 + j10)\ \Omega \tag{5.3}$$

The total equivalent impedance of the circuit is

$$\begin{aligned} \boldsymbol{Z}_{in} = \boldsymbol{Z}_1 + \boldsymbol{Z}_2 \times \boldsymbol{Z}_3 &= -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8} \\ = -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} &= -j10 + 3.22 - j1.07 \\ &= (3.22 - j11.07)\ \Omega \end{aligned} \tag{5.4}$$

5.2 AC circuit analysis

Determine the $v_0(t)$ voltage of the circuit shown in Fig. 5.2 when the source voltage is $v_S(t) = 20\cos(4t - 15°)\ V$

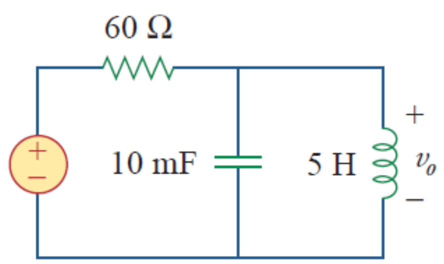

Figure 5.2 Circuit for AC analysis

Solution

Let's transform the source voltage from time domain to frequency (or phasor) domain.

$$v_S(t) = 20\cos(4t - 15°) \rightarrow \boldsymbol{V}_S = 20\, e^{-j15°},\ (\omega = 4) \tag{5.5}$$

To impedance calculation we need the reactance of the capacitor is

$$X_C = \frac{1}{\omega C} = 25\,\Omega \rightarrow \boldsymbol{Z}_C = -j25\,\Omega \tag{5.6}$$

and the reactance of the inductor is

$$X_L = \omega L = 20\,\Omega \rightarrow \boldsymbol{Z}_L = j20\,\Omega \tag{5.7}$$

The impedance of the parallel capacitor and inductor is

$$\boldsymbol{Z}_{LC} = \boldsymbol{Z}_L \times \boldsymbol{Z}_C = \frac{j20 \cdot (-j25)}{-j5} = j100\,\Omega \tag{5.8}$$

Now we can calculate the complex $\boldsymbol{V}_0$ voltage using voltage division

$$\boldsymbol{V}_0 = \boldsymbol{V}_S \frac{\boldsymbol{Z}_{LC}}{60 + \boldsymbol{Z}_{LC}} = 20\, e^{-j15°} \frac{j100}{60 + j100} = \cdots = 17.15\, e^{j15.96°} \tag{5.9}$$

from which result the time varying function of v_0 is

$$v_0(t) = 17.15\cos(4t + 15.96°)\ V \tag{5.10}$$

Alternatively, we could use i.e. nodal analysis also for $\boldsymbol{V}_0$ calculation.

$$\frac{\boldsymbol{V}_S - \boldsymbol{V}_0}{60} = \frac{\boldsymbol{V}_0 - 0}{-j25} + \frac{\boldsymbol{V}_0 - 0}{j20} \rightarrow \cdots \rightarrow \boldsymbol{V}_0 = 17.15\, e^{j15.96°} \tag{5.11}$$

5.3 AC nodal analysis

Find i_x in the circuit shown in Fig. 5.3. The source voltage is $v_S(t) = 20\cos 4t\ V$

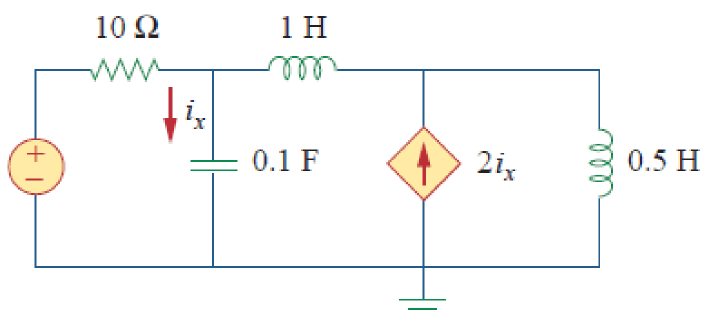

Figure 5.3 Example circuit for nodal analysis

Solution

To transform the circuit form time domain to frequency domain we need the impedance parameters of each circuit element on angular frequency of 4 rad/s.

$$\boldsymbol{Z}_C = -j\frac{1}{\omega C} = -j2.5\ \Omega \tag{5.12}$$

$$\boldsymbol{Z}_{L1} = j\omega L_1 = j4\ \Omega \tag{5.13}$$

$$\boldsymbol{Z}_{L2} = j\omega L_2 = j2\ \Omega \tag{5.14}$$

The complex voltage of the source is

$$\boldsymbol{V}_S = 20\ V \tag{5.15}$$

The circuit in frequency domain is given in Fig. 5.4.

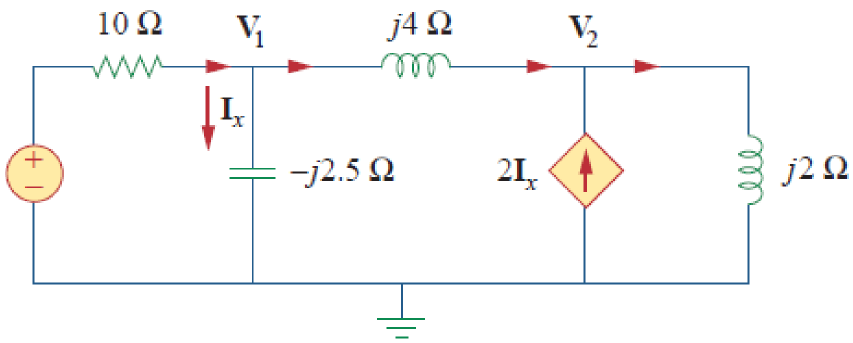

Figure 5.4 The circuit in frequency domain

The nodal equation at node 1 (voltage V_1) is like this

$$\frac{20-\boldsymbol{V}_1}{10} = \frac{\boldsymbol{V}_1}{-j2.5} + \frac{\boldsymbol{V}_1-\boldsymbol{V}_2}{j4} \rightarrow (1+j1.5)\boldsymbol{V}_1 + j2.5\boldsymbol{V}_2 = 20 \tag{5.16}$$

The nodal equation at node 2 (voltage V_2) and the Ohm's on the 0.1-F capacitor is

$$2\boldsymbol{I}_x + \frac{\boldsymbol{V}_1-\boldsymbol{V}_2}{j4} = \frac{\boldsymbol{V}_2}{j2} \quad (\&) \quad \boldsymbol{I}_x = \frac{\boldsymbol{V}_1}{-j2.5} \rightarrow 11\boldsymbol{V}_1 + 15\boldsymbol{V}_2 = 0 \tag{5.17}$$

(5.17) and (5.17) in matrix form

$$\begin{bmatrix} 1+j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix} \tag{5.18}$$

The solution, using Cramer's rule

$$\Delta = \begin{vmatrix} 1+j1.5 & j2.5 \\ 11 & 15 \end{vmatrix} = 15-j5 \tag{5.19}$$

$$\Delta_1 = \begin{vmatrix} 20 & j2.5 \\ 0 & 15 \end{vmatrix} = 300 \tag{5.20}$$

$$\Delta_2 = \begin{vmatrix} 1 + j1.5 & 20 \\ 11 & 0 \end{vmatrix} = -220 \tag{5.21}$$

Thus,

$$\boldsymbol{V}_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15 - j5} = 18.97\, e^{j18.43°} \tag{5.22}$$

$$\boldsymbol{V}_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15 - j5} = 13.91\, e^{j198.3°} \tag{5.23}$$

and

$$\boldsymbol{I}_x = \frac{\boldsymbol{V}_1}{-j2.5} = \frac{18.97\, e^{j18.43°}}{2.5\, e^{-j90°}} = 7.59\, e^{j108.4°} \tag{5.24}$$

So, i_x current in time domain can be written as in (5.25)

$$i_x = 7.59 \cos(4t + 108.4°)\ A \tag{5.25}$$

5.4 AC mesh analysis

Determine current $\mathbf{I}_o$ in the circuit of Fig. 5.5.

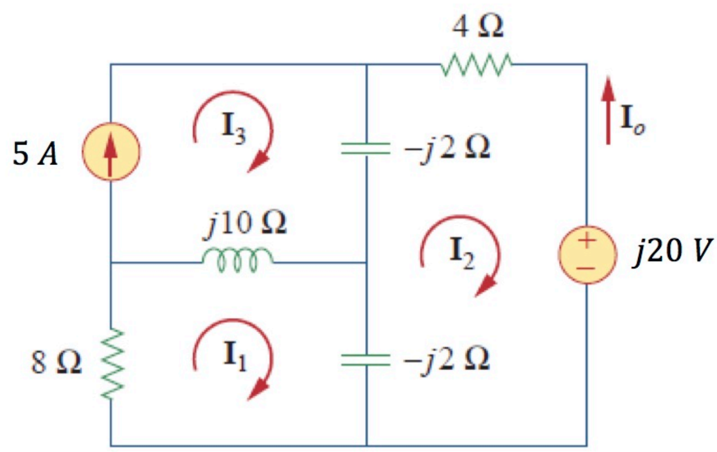

Figure 5.5 Circuit example for mesh analysis

Solution

Applying KVL we can write three mesh equations according to Fig. 5.5.

$$(8 + j10 - j2)\boldsymbol{I}_1 - (-j2)\boldsymbol{I}_2 - j10\boldsymbol{I}_3 = 0 \tag{5.26}$$

$$(4 - j2 - j2)\boldsymbol{I}_2 - (-j2)\boldsymbol{I}_1 - (-j2)\boldsymbol{I}_3 + j20 = 0 \tag{5.27}$$

$$\boldsymbol{I}_3 = 5 \tag{5.28}$$

Substituting (5.28) into (5.26) and (5.27) we get

$$(8 + j8)\boldsymbol{I_1} + j2\boldsymbol{I}_2 = j50 \tag{5.29}$$

$$j2\boldsymbol{I_1} + (4 - j4)\,\boldsymbol{I}_2 = -j20 - j10 \tag{5.30}$$

Written (5.29) and (5.30) in matrix form

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix} \tag{5.31}$$

and calculating determinants for applying Cramer's rule in solution

$$\Delta = \begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix} = 32(1 + j)(1 - j) + 4 = 68 \tag{5.32}$$

$$\Delta_2 = \begin{vmatrix} 8 + j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\, e^{-j35.22} \tag{5.33}$$

Thus,

$$\boldsymbol{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\, e^{-j35.22}}{68} = 6.12\, e^{-j35.22} \tag{5.34}$$

$$\boldsymbol{I}_0 = -\boldsymbol{I}_2 = 6.12\, e^{j144.78} \tag{5.35}$$

5.5 Superposition theorem in AC circuits

Find $\boldsymbol{I_o}$ in the circuit of Fig. 5.5 using superposition theorem.

Solution

According to superposition theorem we take into consideration sources in the circuit separately and $\boldsymbol{I_0}$ will be calculated as addition (superposition) of the part results as shown in (5.36)

$$\boldsymbol{I}_0 = \boldsymbol{I}_0' + \boldsymbol{I}_0'' \tag{5.36}$$

The effect of the voltage source will be calculated on the circuit shown in Figure 5.6.

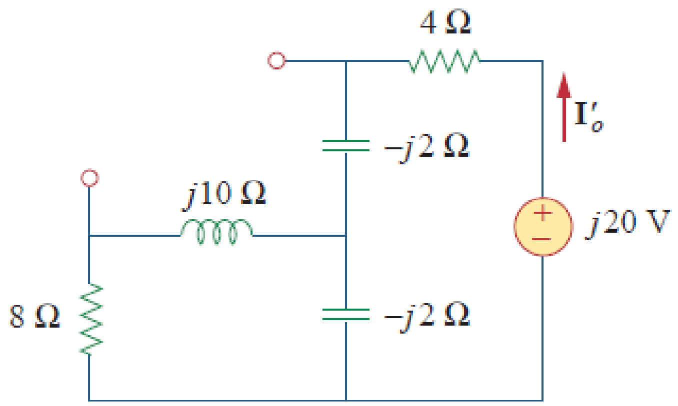

Figure 5.6 Effect of the voltage source

The impedance of the series 8-Ω resistor and j10-Ω inductor parallel to the –j2-Ω capacitor is

$$Z = \frac{-j2(8+j10)}{-j2+8+j10} = 0.25 - \mathrm{j}2.25 \tag{5.37}$$

The part impedance calculated in (5.37I is connected series to the 4-Ω resistor and –j2-Ω capacitor so the electric current can be calculated as

$$I'_0 = \frac{j20}{4-j2+Z} = \frac{j20}{4.25-j4.25} = -2.353 + j2.353 \tag{5.38}$$

The effect of the current source will be calculated on the circuit shown in Figure 5.7.

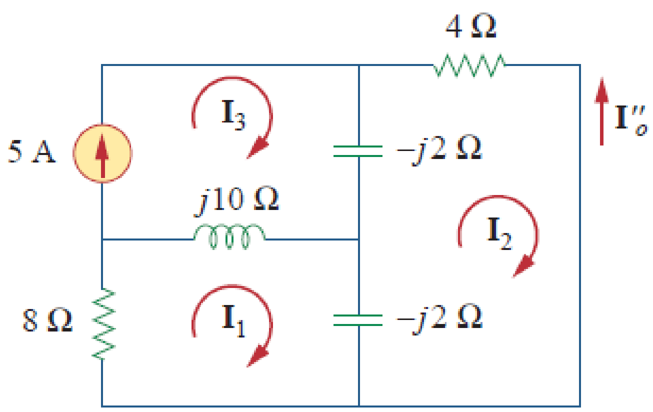

Figure 5.7 Effect of the current source

The mesh equations of the circuit are

$$(8+j8)I_1 - j10I_3 + j2I_2 = 0 \tag{5.39}$$

$$(4-j4)I_2 + j2I_1 + j2I_3 = 0 \tag{5.40}$$

$$I_3 = 5 \tag{5.41}$$

Substituting (5.41) into (5.39) and (5.40)

$$(4-j4)\boldsymbol{I}_2 + j2\boldsymbol{I}_1 + j10 = 0 \rightarrow \boldsymbol{I}_1 = (2+j2)\boldsymbol{I}_2 - 5 \tag{5.42}$$

$$(8+j8)[(2+j2)\boldsymbol{I}_2 - 5] - j50 + j2\boldsymbol{I}_2 = 0 \tag{5.43}$$

Thus,

$$\boldsymbol{I}_2 = \frac{90-j40}{34} = 2.647 - j1.176 \tag{5.44}$$

But I_2 is opposite to I_0'', so

$$\boldsymbol{I}_0'' = -\boldsymbol{I}_2 = -2.647 + j1.176 \tag{5.45}$$

Finally, we calculate I_0 as superposition of the part results

$$\boldsymbol{I}_0 = \boldsymbol{I}_0' + \boldsymbol{I}_0'' = -2.353 + j2.353 - 2.647 + j1.176 \tag{5.46}$$

The I_0 electric current is

$$\boldsymbol{I}_0 = -5 + \mathrm{j}3.529 = 6.12\, e^{j144.78°}\, A \tag{5.47}$$

5.6 AC source transformation

Calculate V_x in the circuit shown in Figure 5.8.

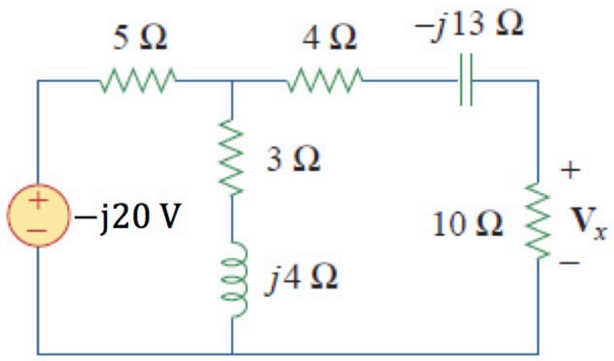

Figure 5.8 Circuit for source transformation example

To solve the problem, we use source transformation to simplify the structure of the circuit so, we transform the Thevenin generator to Norton generator a first. Then we calculate Z_1 as shown in Fig. 5.9.

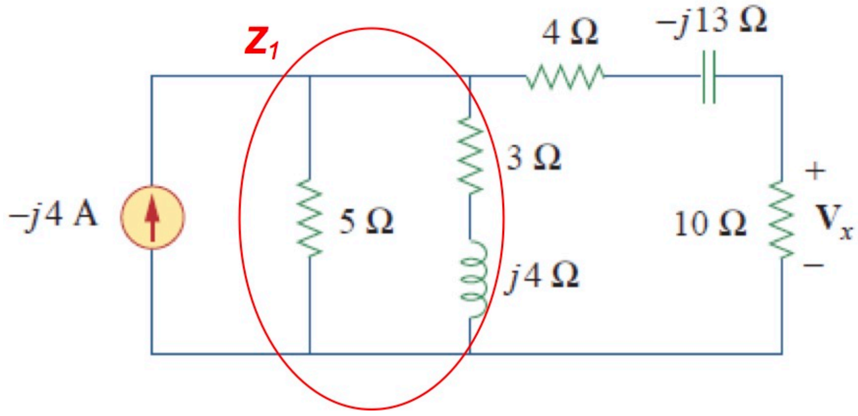

Figure 5.9

$$Z_1 = \frac{5(3+j4)}{8+j4} = 2.5 + \mathrm{j}1.25\ \Omega \tag{5.48}$$

Then, we transform the Norton generator to its Thevenin equivalent. Thevenin impedance is the same as the Norton impedance. Thevenin voltage is calculated as (5.49).

$$V_S = I_S \cdot Z_1 = -j4(2.5 + \mathrm{j}1.25) = 5 - j10\ \mathrm{V} \tag{5.49}$$

The result is shown in Fig. 5.10.

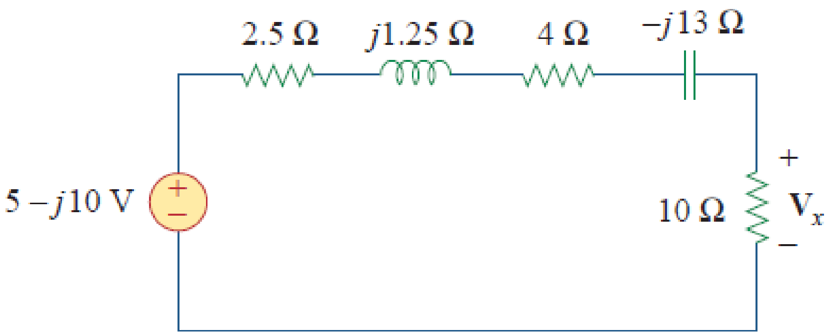

Figure 5.10 The circuit after source transformations

Applying voltage division for the circuit in Fig. 5.10

$$V_x = (5 - j10)\frac{10}{10 + 2.5 + j1.25 + 4 - j13} = 5.519\ e^{-j28°}\ \mathrm{V} \tag{5.50}$$

5.7 Thevenin equivalent in AC circuits

Find the Thevenin equivalent of the circuit in Fig 5.11 as seen from terminals a-b.

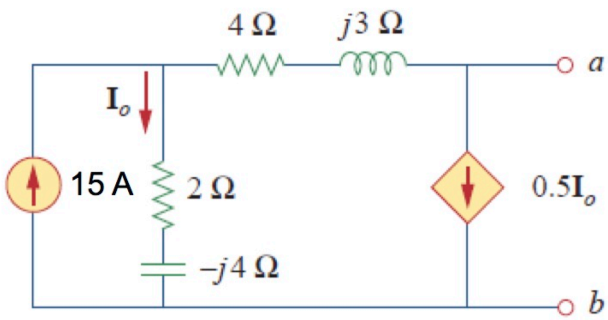

Figure 5.11 Circuit for Thevenin equivalent calculation

Solution

We have to determine the Thevenin voltage and Thevenin impedance.

To obtain V_{Th} we use the circuit in Fig. 5.12.

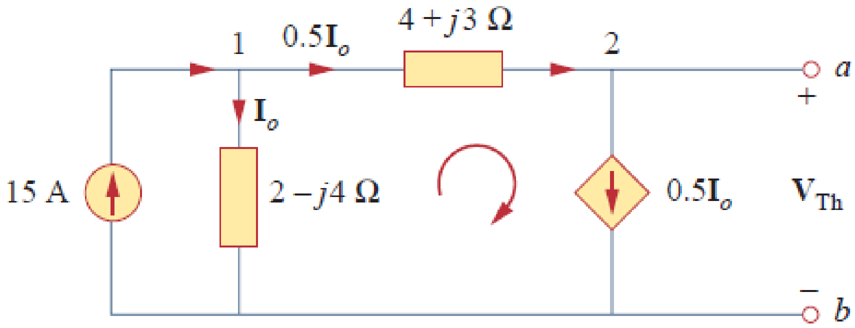

Figure 5.12 Finding Thevenin voltage

Applying KCL for node 1

$$15 = \boldsymbol{I}_0 + 0.5\boldsymbol{I}_0 \rightarrow \boldsymbol{I}_0 = 10\ A \tag{5.51}$$

Applying KVL for the signed mesh

$$-\boldsymbol{I}_0(2 - j4) + 0.5\boldsymbol{I}_0(4 + j3) + V_{Th} = 0 \tag{5.52}$$

from which

$$\boldsymbol{V}_{Th} = 10(2 - j4) - 5(4 + j3) = -j55 \tag{5.53}$$

To obtain $\boldsymbol{Z}_{Th}$, we remove the **independent** source *(only!)* as seen in Fig. 5.13.

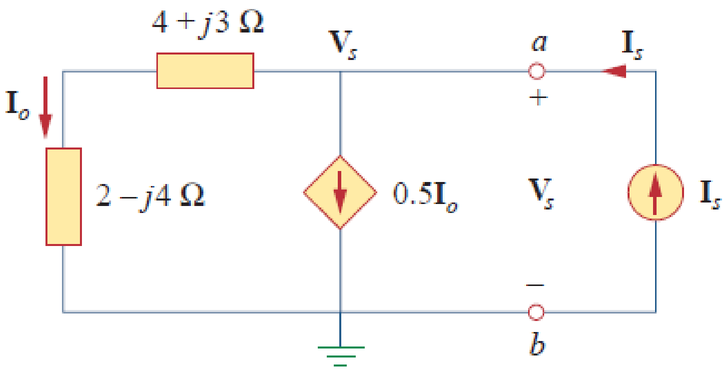

Figure 5.13 Finding Thevenin impedance

Applying KCL for the upper node

$$\boldsymbol{I}_S = \boldsymbol{I}_0 + 0.5\boldsymbol{I}_0 \rightarrow \boldsymbol{I}_0 = \frac{\boldsymbol{I}_S}{1.5} \tag{5.54}$$

For convenience, we choose I_s = 3 A and in this case I_0 = 2 A. Because of Ohm's law we can write (5.55).

$$\boldsymbol{V}_S = \boldsymbol{I}_0(4 + j3 + 2 - j4) = 2(6 - j) \tag{5.55}$$

Thus,

$$\boldsymbol{Z}_{Th} = \frac{\boldsymbol{V}_S}{\boldsymbol{I}_S} = \frac{2(6 - j)}{3} = 4 - j0.67\ \Omega \tag{5.56}$$

So, the Thevenin equivalent is determined by (5.53) and (5.56).

5.8 Norton equivalent in AC circuits

Obtain current I_o in Fig. 5.14 using Norton's theorem.

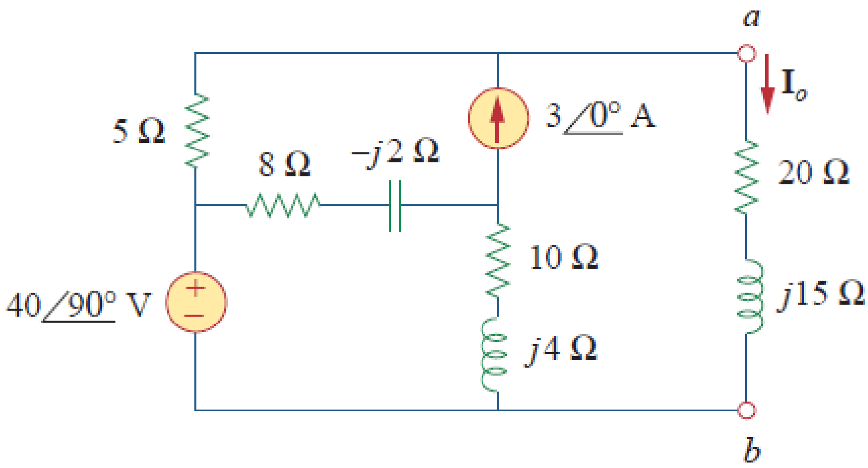

Figure 5.14 Circuit for Norton equivalent calculation

To obtain Z_N we can use Fig. 5.15 that shows that Norton impedance is simply 5-Ω.

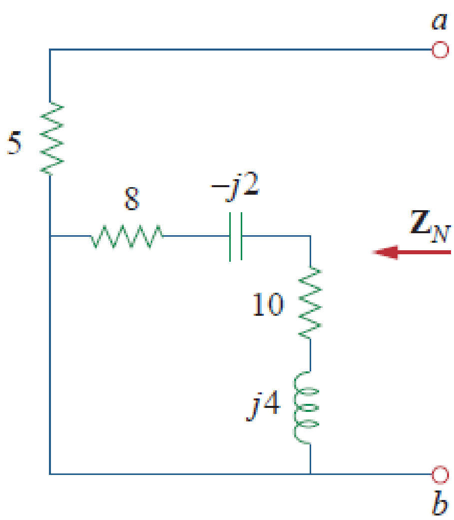

Figure 5.15 Finding Norton impedance

To get the Norton current we use the circuit shown in Fig. 5.16.

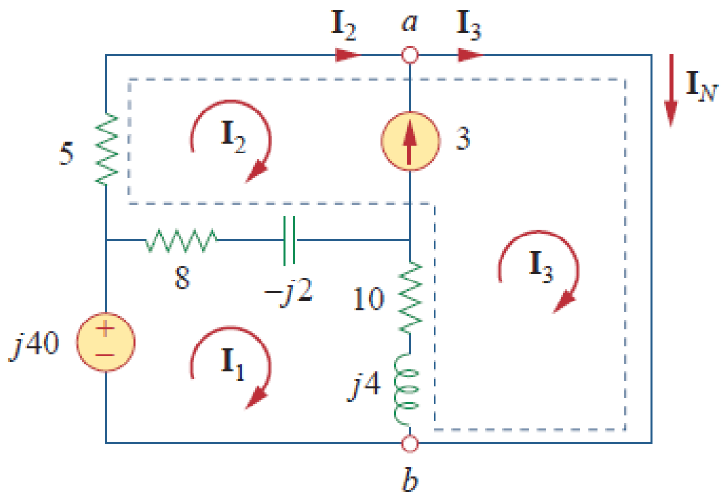

Figure 5.16 Finding Norton current

Because the circuit contains a super mesh we can write two mesh equations

$$-j40 + (18 + j2)\boldsymbol{I}_1 - (8 - j2)\boldsymbol{I}_2 - (10 + j4)\boldsymbol{I}_3 = 0 \tag{5.57}$$

$$(13 - j2)\boldsymbol{I}_2 + (10 + j4)\boldsymbol{I}_3 - (18 + j2)\boldsymbol{I}_1 = 0 \tag{5.58}$$

and the super mesh condition equation

$$\boldsymbol{I}_3 = \boldsymbol{I}_2 + 3 \tag{5.59}$$

Adding (5.57) and (5.58) equations we have (5.60)

$$-j40 + 5\boldsymbol{I}_2 = 0 \rightarrow \boldsymbol{I}_2 = j8 \tag{5.60}$$

Substituting (5.60) into (5.59)

$$\boldsymbol{I}_N = \boldsymbol{I}_3 = \boldsymbol{I}_2 + 3 = (3 + j8)\ A \tag{5.61}$$

The equivalent circuit of Fig. 5.14 is shown in Fig. 5.17.

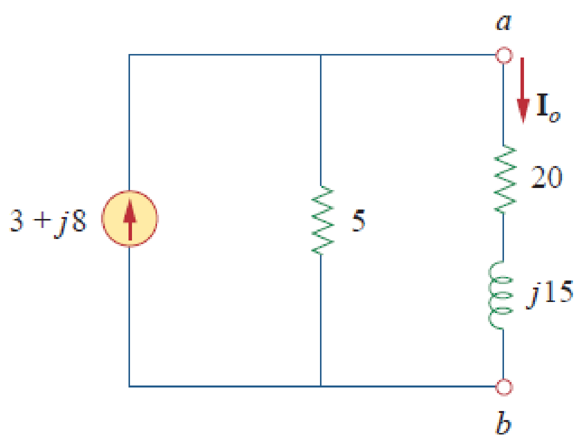

Figure 5.17 Equivalent circuit

Applying current division for the circuit in Fig. 5.17

$$\boldsymbol{I}_0 = \frac{5}{5 + 20 + j15}\boldsymbol{I}_N = \frac{3 + j8}{5 + j3} = \cdots = 1.46\ e^{j38.49^\circ}\ A \tag{5.62}$$

6. AC Power Analysis

6.1 Mean values of sinusoidal signals

Determine the mean values of a $v(t) = V_p \cos \omega t$ voltage.

Solution

According to definition the simple mean can be calculated as given in (6.1).

$$V_{avg} = \frac{1}{T}\int_0^T v(t)dt = \frac{V_p}{T}\int_0^T \cos \omega t \;\; dt = 0 \tag{6.1}$$

The absolute mean value is

$$V_{abs} = \frac{1}{T}\int_0^T |v(t)|dt = \frac{V_p}{T/4}\int_0^{T/4} \cos \omega t \, dt = \frac{4V_p}{T}\left|\frac{\sin \omega t}{\omega}\right|_0^{T/4} = \frac{4V_p}{T}\frac{T}{2\pi} = \frac{2}{\pi}V_p \tag{6.2}$$

and the RMS or effective value is

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} = \sqrt{\frac{4V_p^2}{T}\int_0^T cos^2 \omega t dt} \tag{6.3}$$

By applying the following trigonometric formula of (6.4) for (6.3) we can write (6.5)

$$\begin{gathered} cos\, 2 \propto = cos^2 \propto - sin^2 \propto \\ cos^2 \propto + sin^2 \propto = 1 \\ cos^2 \propto = \frac{1 + cos 2\omega t}{2} \end{gathered} \tag{6.4}$$

$$= \sqrt{\frac{4V_p^2}{T}\int_0^{T/4} \frac{1 + cos 2\omega t}{2} dt} = \sqrt{\frac{4V_p^2}{T}\left[\frac{1}{2}t + \frac{sin 2\omega t}{4\omega}\right]_0^{T/4}} = \sqrt{\frac{V_p^2}{2}} = \frac{V_p}{\sqrt{2}} \tag{6.5}$$

Note the sinusoidal signal

$$\frac{V_{rms}}{V_{abs}} = \frac{\pi}{2\sqrt{2}} \cong 1.11 \tag{6.6}$$

6.2 Average power

Find the average power supplied by the source and the average power absorbed by the resistor in the circuit of Fig. 6.1.

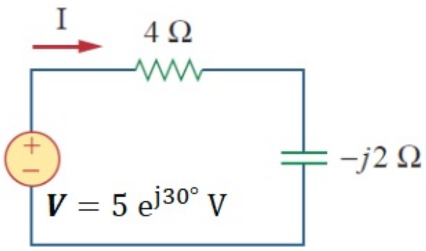

Figure 6.1 Circuit for AC power calculation

Solution

We need the electric current for the power calculation that is

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}} = \frac{5\,e^{j30°}}{4-j2} = \frac{5\,e^{j30°}}{4.472\,e^{-j26.57°}} = 1.118\,e^{j56.57°} A \qquad (6.7)$$

The average power of the source is

$$P = V \cdot I \cdot cos(\varphi_U - \varphi_I) = 5 \cdot 1.118 \cdot cos(30° - 56.57°) = 5\,W \qquad (6.8)$$

The voltage across the resistor for its power calculation is

$$\boldsymbol{V}_R = 4 \cdot \boldsymbol{I} = 4.472\,e^{j56.57°}\,V \qquad (6.9)$$

and the dissipated power on the resistor is

$$P_R = V_R \cdot I = 4.472 \cdot 1.118 = 5\,W \qquad (6.10)$$

Note that we can verify the result as the dissipated power on the resistor must be equal to the real part of the total complex power in the circuit.

$$P = Re\{\boldsymbol{S}\} = Re\{\boldsymbol{V} \cdot \boldsymbol{I}^*\} = \cdots = 5\,W \qquad (6.10)$$

6.3 Maximum power transfer

Determine Z_L in the circuit shown on Fig. 6.2 to maximize the average power acting on it. Calculate the value of this maximum average power.

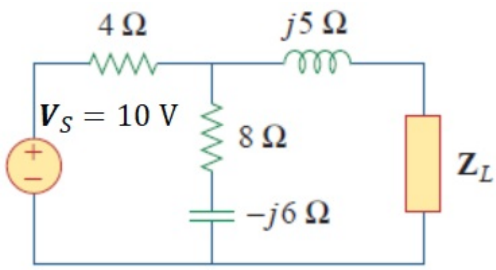

Figure 6.2 Circuit for maximum power transfer calculation

Solution

To find the optimal load impedance for maximum power transfer we need to find the Thevenin equivalent of the active circuit connected to the load impedance. To determine the Thevenin impedance we can use the circuit in Fig. 6.3.

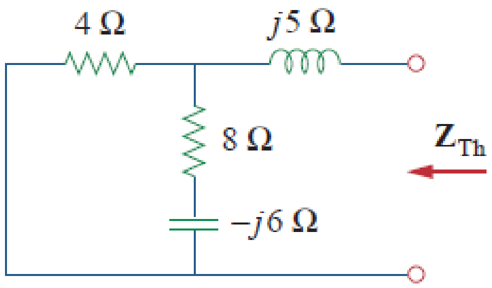

Figure 6.3 Finding Thevenin impedance

The equivalent impedance of the circuit in Fig. 6.3 is

$$\boldsymbol{Z}_{Th} = \mathrm{j}5 + 4 \times (8 - j6) = j5 + \frac{4(8-j6)}{12-j6} = (2.933 + j4.467)\ \Omega \tag{6.11}$$

For calculating Thevenin voltage we use Fig. 6.4.

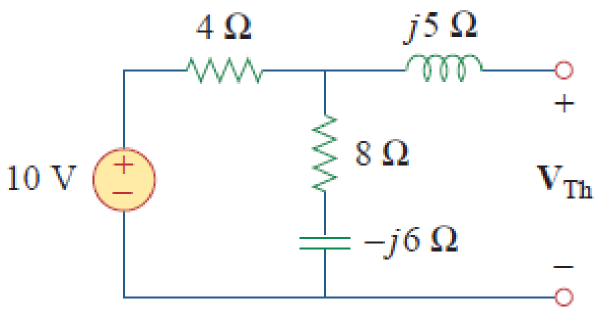

Figure 6.4 Circuit for calculating V_{Th}

Applying voltage division for the circuit

$$\boldsymbol{V}_{Th} = 10\frac{8-j6}{4+8-j6} = 7.454\ e^{-j10.3°}\ V \tag{6.12}$$

The condition of maximum power transfer if the load impedance is equal to the conjugate of the Thevenin impedance is

$$\boldsymbol{Z}_L = \boldsymbol{Z}^*_{Th} = (2.933 - j4.467)\ \Omega \tag{6.13}$$

The maximum power can be considered in this case as in (6.14)

$$P_{max} = \frac{|\boldsymbol{V}_{Th}|^2}{8R_{Th}} = \frac{(7.454)^2}{8 \cdot 2.933} = 2.368\ \mathrm{W} \tag{6.14}$$

6.4 Complex power

Find the complex and apparent powers, the real and reactive powers, the power factor and load impedance if $v(t) = 60\,cos(\omega t - 10°)\ V,\ i(t) = 1.5\cos(\omega t + 50°)\ A$

Solution

Complex voltage and current are shown in (6.15)

$$\boldsymbol{V}_{rms} = \frac{60}{\sqrt{2}}\, e^{-j10^\circ}\,, \qquad \boldsymbol{I}_{rms} = \frac{1.5}{\sqrt{2}}\, e^{j50^\circ} \tag{6.15}$$

from which the complex power, according its definition is

$$\boldsymbol{S} = \boldsymbol{V}_{rms}\boldsymbol{I}^*_{rms} = \frac{60}{\sqrt{2}}\, e^{-j10^\circ} \cdot \frac{1.5}{\sqrt{2}}\, e^{-j50^\circ} = 45\, e^{-j60^\circ}\, VA \tag{6.16}$$

Apparent power is an absolute value of the complex power

$$S = 45\, VA \tag{6.17}$$

To find the real and reactive powers we have to transform complex power from polar to algebraic form.

$$\boldsymbol{S} = 45\, e^{-j60^\circ} = 45\cos(-60^\circ) + j45\sin(-60^\circ) = (22.5 - j38.97)\, VA \tag{6.18}$$

The real part is the active (real) power and the imaginary part is reactive power.

$$P = 22.5\, W\,, \qquad Q = -38.97\, VAR \tag{6.19}$$

Finding the power factor

$$pf = \cos(-60^\circ) = 0.5\ (leading = CAP) \tag{6.20}$$

That is the leading (capacitive) power factor because the voltage is delayed to the current. Finally, the load impedance is

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{60\, e^{-j10^\circ}}{1.5\, e^{j50^\circ}} = 40e^{\,-j60^\circ}\ \Omega \tag{6.21}$$

This is a capacitive impedance because of its negative phase.

6.5 Power factor correction

Connected to a 230 V_{RMS}, 50 Hz power line, a load absorbs 4 kW with a lagging power factor of 0.8. Find the value of capacitance necessary to raise the power factor to 0.9.

Solution

The original power factor is given as $\cos\varphi_1 = 0.8$ the apparent power can be determined according to (6.22)

$$S_1 = \frac{P}{cos\,\varphi_1} = \frac{4000}{0.8} = 5000\, VA \tag{6.22}$$

For the reactive power calculation we need the value of $\sin\varphi_1$. According to the trigonometric formula of

$$\begin{gathered} sin^2\alpha + cos^2\alpha = 1 \to \sin\alpha = \sqrt{1 - cos^2\alpha} \\ \sin\alpha = \sqrt{1 - 0.8^2} = 0.6 \end{gathered} \tag{6.23}$$

So, the reactive power is

$$Q_1 = S_1 \, sin\,\varphi_1 = 5000 \cdot 0.6 = 3000\,VAr \tag{6.24}$$

Our goal is to increase the power factor to 0.9 i.e. to reach the apparent power of

$$S_2 = \frac{P}{cos\,\varphi_2} = \frac{4000}{0.9} = 4444.4\,VA \tag{6.25}$$

Finding total reactive power after compensation

$$sin\,\varphi_2 = \sqrt{1 - 0.9^2} = 0.43 \tag{6.26}$$

and

$$Q_2 = S_2 \, sin\,\varphi_2 = 1937.15\,VAr \tag{6.27}$$

To decrease the original reactive power from the value in (6.24) to the value in (6.27) we need a capacitor with the opposite sign of reactive power, that is given in (6.28).

$$Q_C = Q_1 - Q_2 = 3000 - 1937.15 = 1062.85\,VAr \tag{6.28}$$

Because the capacitor is connected in parallel to the load and it only has reactive power, we can write (6.29).

$$Q_C = \frac{V_{RMS}^2}{X_C} \tag{6.29}$$

Expressing the capacitance from (6.29) we get the necessary capacitance for the required compensation of power factor.

$$C = \frac{Q_C}{\omega V_{RMS}^2} = \frac{1062.85}{2\pi \cdot 50 \cdot 230^2} = 63.98\,\mu F \tag{6.29}$$

7. Three-Phase Circuits

7.1 Balanced wye-wye (Y-Y) connection

Calculate the line currents in the three-wire Y-Y system shown in Fig. 7.1.

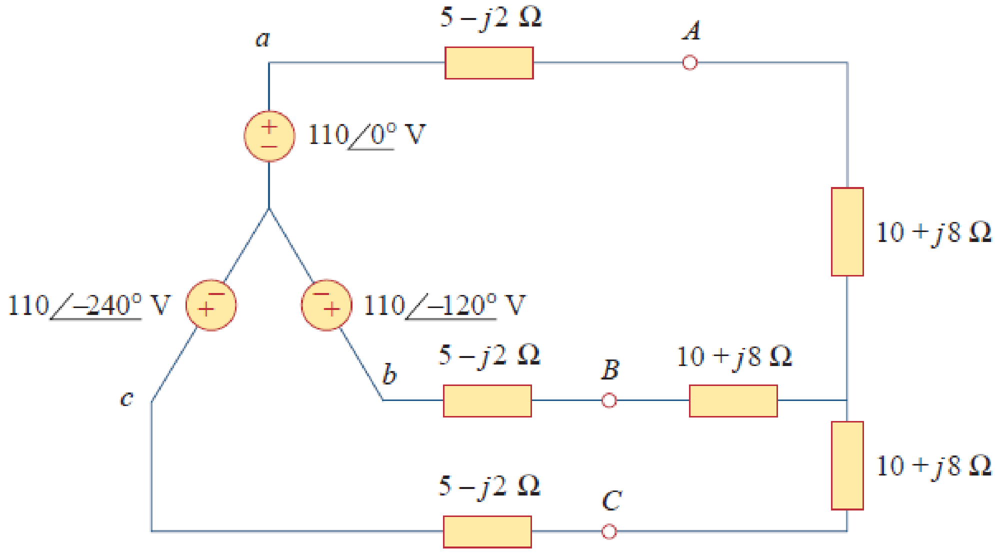

Figure 7.1 Three-wire wye-wye system

Solution

We can use the single phase equivalent circuit shown in Fig. 7.2 for our calculation because system is balanced.

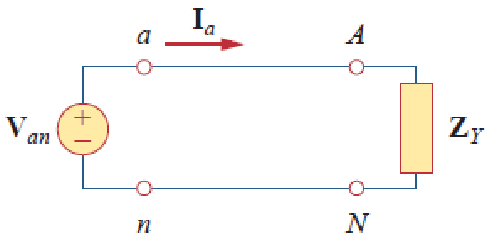

Figure 7.2 Single phase equivalent circuit

The line current in Fig. 7.2 is

$$\boldsymbol{I}_a = \frac{\boldsymbol{V}_{an}}{\boldsymbol{Z}_Y} \tag{7.1}$$

The total load impedance in each line is

$$\boldsymbol{Z}_Y = (5 - j2) + (10 + j8) = 16.16 \cdot e^{j21.8°} \tag{7.2}$$

Thus, the line currents are the following

$$\boldsymbol{I}_a = \frac{110 \cdot e^{j0°}}{16.16 \cdot e^{j21.8°}} = 6.81 \cdot e^{-j21.8°} \tag{7.3}$$

$$\boldsymbol{I}_b = \boldsymbol{I}_a \cdot e^{-j120°} = 6.81 \cdot e^{-j141.8°} \tag{7.4}$$

$$\boldsymbol{I}_c = \boldsymbol{I}_a \cdot e^{-j240°} = 6.81 \cdot e^{j98.2°} \tag{7.5}$$

Note: Only I_a current is calculated using Ohm's law because I_b and I_c line currents have the same values with phase delays of 120° due to the balanced conditions. It is written in (7.4) and (7.5).

7.2 Balanced wye-delta (Y-D) connection

A balanced a-b-c-sequence Y-connected source with $V_{an} = 100\ \ V$ supplies a Δ-connected balanced load 8+j4 Ω per phase. Calculate the phase and line currents.

Solution 1

For convenience we transform the phase impedance from algebraic to polar form.

$$\boldsymbol{Z}_\Delta = 8 + j4 = 8.944e^{j26.57°}\ \Omega \tag{7.6}$$

Once we know the phase voltage on the source side we can obtain a line voltage that gives the phase voltage on the load side hence the load is delta connected.

$$\boldsymbol{V}_{an} = 100\ e^{j10°} \to \boldsymbol{V}_{ab} = \boldsymbol{V}_{an}\sqrt{3}e^{j30°} = \boldsymbol{V}_{AB} = 173.2e^{j40°}\ V \tag{7.7}$$

Thus, the phase current on load is

$$\boldsymbol{I}_{AB} = \frac{\boldsymbol{V}_{AB}}{\boldsymbol{Z}_\Delta} = \frac{173.2e^{j40°}}{8.944e^{j26.57°}} = 19.36e^{j13.43°}\ A \tag{7.8}$$

The load is balanced so the phase currents in phase b and c is given as simple rotation.

$$\boldsymbol{I}_{BC} = \boldsymbol{I}_{AB}e^{-j120°} = 19.36e^{-j106.57°}\ A \tag{7.9}$$

$$\boldsymbol{I}_{CA} = \boldsymbol{I}_{AB}e^{j120°} = 19.36e^{j133.43°}\ A \tag{7.10}$$

And the line currents will be the following.

$$\boldsymbol{I}_a = \boldsymbol{I}_{AB}\sqrt{3}e^{-j30°} = 19.36\sqrt{3}e^{-j16.57°} = 33.53e^{-j16.57°}\ A \tag{7.11}$$

$$\boldsymbol{I}_b = \boldsymbol{I}_a e^{-j120°} = 33.53e^{-j136.57°}\ A \tag{7.12}$$

$$\boldsymbol{I}_c = \boldsymbol{I}_a e^{j120°} = 33.53e^{j103.43°}\ A \tag{7.13}$$

Solution 2

By taking advantage of the balanced condition, we can also use single phase analysis. Knowing that the wye connected equivalent phase impedance is the third part of the delta connected phase impedance, I_a line current is given by Ohm's law in (7.14).

$$\boldsymbol{I}_a = \frac{\boldsymbol{V}_{an}}{\boldsymbol{Z}_\Delta/3} = \frac{100e^{j10°}}{2.981e^{j26.57°}} = 33.546e^{-j16.57°}\ A \tag{7.14}$$

Other line currents are given by (7.12) and (7.13) in the same way, because of the balanced conditions.

7.3 Balanced delta-delta (D-D) connection

A balanced Δ-connected load, having an impedance of $20\text{-j}15\ \Omega$, is connected to a Δ-connected, positive-sequence generator having $V_{ab}=330\ \mathrm{V}$. Calculate the phase currents of the load and the line currents in the circuit.

Solution

For convenience we transform the phase impedance from algebraic to polar form.

$$\boldsymbol{Z}_{\Delta} = 20 - j15 = 25e^{-j36.87°}\ \Omega \tag{7.15}$$

The phase currents in each phase are given in (7.16), (7.17) and (7.18).

$$\boldsymbol{I}_{AB} = \frac{\boldsymbol{V}_{AB}}{\boldsymbol{Z}_{\Delta}} = \frac{330}{25e^{-j36.87°}} = 13.2e^{j36.87°}\ A \tag{7.16}$$

$$\boldsymbol{I}_{BC} = \boldsymbol{I}_{AB}e^{-j120°} = 13.2e^{-j83.13°}\ A \tag{7.17}$$

$$\boldsymbol{I}_{CA} = \boldsymbol{I}_{AB}e^{j120°} = 13.2e^{j156.87°}\ A \tag{7.18}$$

Knowing the phase currents, we obtain the line currents from (7.19), (7.20) and (7.21).

$$\boldsymbol{I}_{a} = \boldsymbol{I}_{AB}\sqrt{3}e^{-j30°} = 13.2\sqrt{3}e^{-j6.87°} = 22.86e^{-j6.87°}\ A \tag{7.19}$$

$$\boldsymbol{I}_{b} = \boldsymbol{I}_{a}e^{-j120°} = 22.86e^{-j113.13°}\ A \tag{7.20}$$

$$\boldsymbol{I}_{c} = \boldsymbol{I}_{a}e^{j120°} = 22.86e^{-j126.87°}\ A \tag{7.21}$$

7.4 Balanced delta-wye (D-Y) connection

A balanced Y-connected load is supplied by a balanced, positive sequence delta–connected source. Calculate the phase currents if $\boldsymbol{Z}_{\mathrm{Y}} = (40 + \mathrm{j}25)\ \Omega$, $\boldsymbol{V}_{\mathrm{ab}} = 210\ V$.

Solution

Once we know the line voltage, the phase voltage is wye connected and the balanced load is given by (7.22).

$$\boldsymbol{V}_{an} = \frac{\boldsymbol{V}_{ab}}{\sqrt{3}} \cdot e^{-j30°} = 121.2 \cdot e^{-j30°} \tag{7.22}$$

Thus, I_a phase current is

$$\boldsymbol{I}_{a} = \frac{\boldsymbol{V}_{an}}{\boldsymbol{Z}_{Y}} = \frac{121.2 \cdot e^{-j30°}}{47.12 \cdot e^{j32°}} = 2.57 \cdot e^{-j62°} \tag{7.23}$$

From which other phase currents are

$$\boldsymbol{I}_{b} = \boldsymbol{I}_{a} \cdot e^{-j120°} = 2.57 \cdot e^{-j182°} \tag{7.24}$$

$$\boldsymbol{I}_{c} = \boldsymbol{I}_{a} \cdot e^{j120°} = 2.57 \cdot e^{j58°} \tag{7.25}$$

7.5 Three-phase power in balanced system

Determine the total average power, reactive power, and complex power at the source and on the load in the circuit shown in Fig. 7.3.

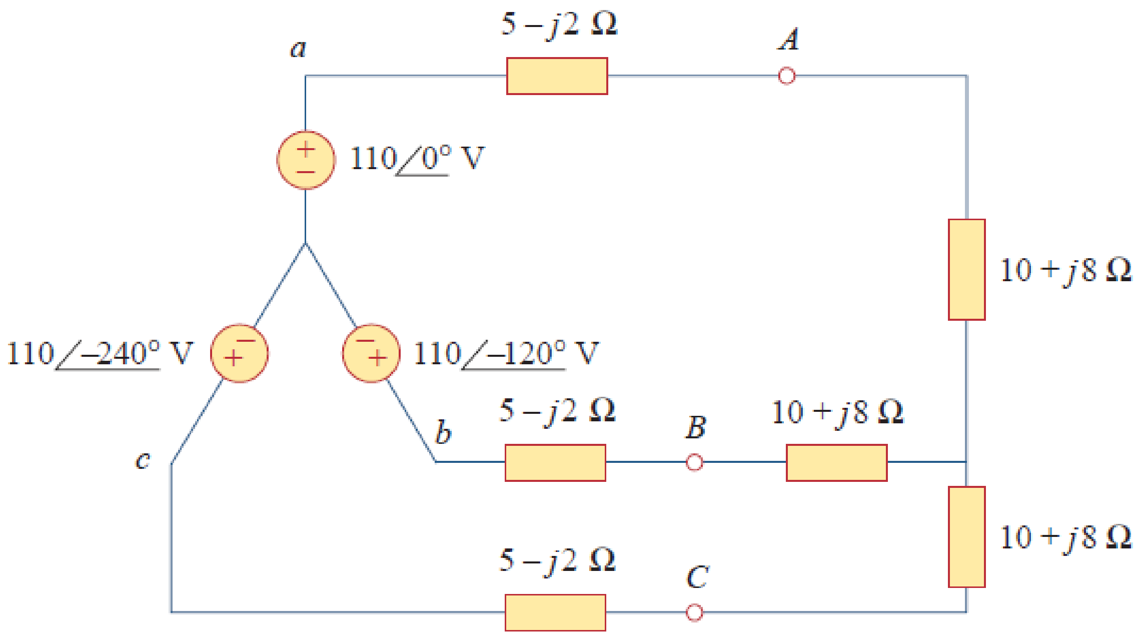

Figure 7.3 Balanced three-phase circuit

Solution

According to Fig. 7.3 the phase voltage in phase a is given as (7.26).

$$\boldsymbol{V}_a = 110\, V \tag{7.26}$$

The equivalent impedance in each phase is built by the series connected line impedance and load impedance. It is calculated in (7.27)

$$\boldsymbol{Z}_Y = (5 - j2) + (10 + j8) = 16.16 \cdot e^{j21.8^\circ}\, \Omega \tag{7.27}$$

Thus, the line current can be calculated in (7.28).

$$\boldsymbol{I}_a = \frac{110}{16.16 \cdot e^{j21.8^\circ}} = 6.81 \cdot e^{-j21.8^\circ} \tag{7.28}$$

Knowing the phase voltages and phase currents on the source side (7.29) gives the total complex power of the three-phase generators.

$$\boldsymbol{S}_S = -3\boldsymbol{V}_a\boldsymbol{I}_a^* = -3 \cdot 110 \cdot 6.81 \cdot e^{j21.8^\circ} = -(2087 + j834.6)\, VA \tag{7.29}$$

Thus, the real part of this complex power gives the active power and the imaginary part gives the reactive power.

$$P_S = -2087\, W\,, \qquad Q_S = -834.6\, VAR \tag{7.30}$$

We need load impedance to calculate the power in polar form. This is given in (7.31).

$$\boldsymbol{Z}_a = 10 + j8 = 12.81 \cdot e^{j38.66^\circ}\, \Omega \tag{7.31}$$

This, the total complex power of the balanced load is obtained from (7.32).

$$\boldsymbol{S}_L = 3\, I_a^2\, \boldsymbol{Z}_a = 3 \cdot 6.81^2 \cdot 12.81 \cdot e^{j38.66^\circ} \tag{7.32}$$

Calculating the real and imaginary part of this complex power we get (7.33)

$$\boldsymbol{S}_L = 1782 \cdot e^{j38.66^\circ} = (1392 + j1113)\, VA \tag{7.33}$$

From which the total active and reactive powers on the load are

$$P_L = 1392\, W\,, \qquad Q_L = 1113\, VAR \tag{7.34}$$

The complex power in the transmission line is given by (7.35).

$$\boldsymbol{S}_l = 3\, I_a^2\, \boldsymbol{Z}_l = 3 \cdot 6.81^2 \cdot (5 - j2) = (695.6 - j278.3)\, VA \tag{7.35}$$

According to Tellegen's theorem the sum of the powers on the generator, transmission line and load side must be equal to zero. If it is then that verifies our calculations.

$$\boldsymbol{S}_S + \boldsymbol{S}_L + \boldsymbol{S}_l = 0 \tag{7.36}$$

7.6 Three-phase system with an unbalanced load

The unbalanced Y-load, shown in Fig. 7.4, has been supplied by the balanced phase voltage system of 230 V phase voltages with the 'a-b-c' sequence. Calculate the line currents and the neutral current. $Z_A = 30\,\Omega,\ Z_B = (15 + j10)\,\Omega,\ Z_C = (10 - j10)\,\Omega$.

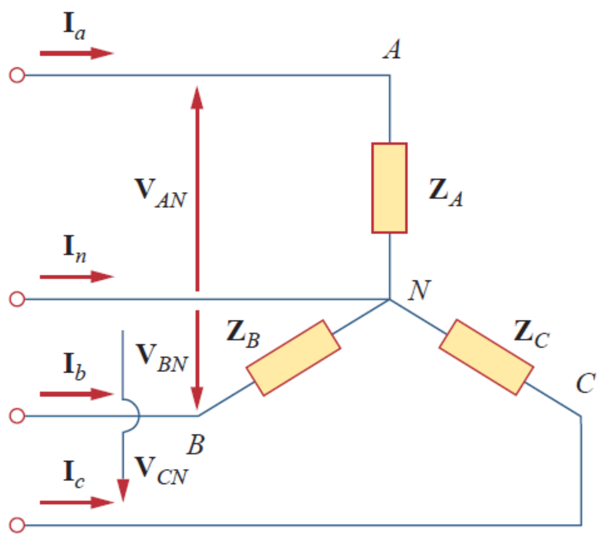

Figure 7.4 An unbalanced three-phase load

Solution

The voltage system is balanced and we can express the phase voltages in complex form.

$$\boldsymbol{V}_{AN} = 230 \cdot e^{j0^\circ}\, V,\ \ \boldsymbol{V}_{BN} = 230 \cdot e^{-j120^\circ}\, V,\ \ \boldsymbol{V}_{CN} = 230 \cdot e^{j120^\circ}\, V \tag{7.37}$$

Applying (generalized) Ohm's law the phase currents are the following. Because the load system is unbalanced it is not enough to calculate only one phase current and apply rotation on it. Each phase current has to be calculated separately.

$$\boldsymbol{I}_a = \frac{\boldsymbol{V}_{AN}}{\boldsymbol{Z}_A} = \frac{230}{30} = 7.67\, A \tag{7.38}$$

The current in phase 'b' is the following.

$$\boldsymbol{I}_b = \frac{\boldsymbol{V}_{BN}}{\boldsymbol{Z}_B} = \frac{230 \cdot e^{-j120°}}{15 + j10} \tag{7.39}$$
$$= \frac{230 \cdot e^{-j120°}}{18.03 \cdot e^{j33.69°}} = 12.75 \cdot e^{-j153.69°} = (-11.42 - j5.65)\ A$$

And the current in phase 'c' is

$$\boldsymbol{I}_c = \frac{\boldsymbol{V}_{CN}}{\boldsymbol{Z}_C} = \frac{230 \cdot e^{j120°}}{10 - j10} \tag{7.40}$$
$$= \frac{230 \cdot e^{j120°}}{14.14 \cdot e^{-j45°}} = 16.27 \cdot e^{j165°} = (-15.71 + j4.21)\ A$$

Finally, applying KVL at node 'N' the neutral current is the following.

$$\boldsymbol{I}_n = -(\boldsymbol{I}_a + \boldsymbol{I}_b + \boldsymbol{I}_c) = (19.46 + j1.44)\ A \tag{7.41}$$

7.7 Potential shift of neutral point

A balanced voltage system supplies the unbalanced load. The phase voltage of the source system is 230 V with a phase sequence of 'a-b-c'. Take $\boldsymbol{V}_{an}$ as a reference and find the $\boldsymbol{V}_{Nn}$ voltage if $Z_A = 10\ \Omega,\ Z_B = j20\ \Omega,\ Z_C = -j10\ \Omega,\ Z_n = 10\ \Omega$.

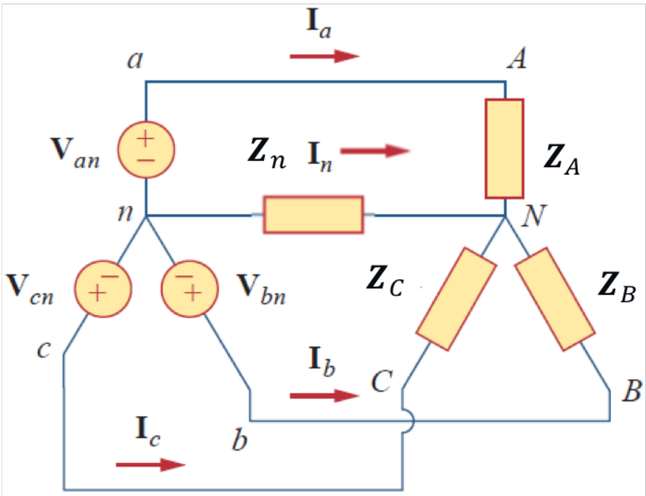

Figure 7.5 Three-phase system with a non-ideal neutral wire

Solution

A convenient way for calculating neutral voltage between the neutral nodes at source and load side is applying Millmann's theorem, given in (7.42).

$$\boldsymbol{V}_{Nn} = \frac{\boldsymbol{V}_{an} \cdot \boldsymbol{Y}_A + \boldsymbol{V}_{bn} \cdot \boldsymbol{Y}_B + \boldsymbol{V}_{cn} \cdot \boldsymbol{Y}_C}{\boldsymbol{Y}_A + \boldsymbol{Y}_B + \boldsymbol{Y}_C + \boldsymbol{Y}_n} = \cdots \tag{7.42}$$

In this equation, admittances are used instead of impedances, so we calculate these elements as the reciprocal of impedances. Results are given in (7.43).

$$\boldsymbol{Y}_A = 0.1\ \mathrm{S}, \qquad \boldsymbol{Y}_B = -\mathrm{j}0.05\ \mathrm{S}, \qquad \boldsymbol{Y}_C = \mathrm{j}0.1\ S, \qquad \boldsymbol{Y}_n = 0.1\ S \tag{7.43}$$

Phase voltages of the balanced voltage system are expressed in (7.44).

$$\boldsymbol{V}_{an} = 230 \cdot e^{j0°}\ V, \qquad \boldsymbol{V}_{bn} = 230 \cdot e^{-j120°}\ V, \qquad \boldsymbol{V}_{cn} = 230 \cdot e^{j120°}\ V \tag{7.44}$$

Substituting voltage and admittance parameters into (7.42) the final result is given by (7.45)

$$\boldsymbol{V}_{Nn} = (-39.41 - j18.92)\,V \rightarrow V_{Nn} = 43.71\,V \tag{7.45}$$

The result in (7.45) shows the neutral potential shift is 43.71 V meaning that neutral point at the load side has a reasonable voltage with respect to the generators neutral point even if the neutral wire exists with a non-zero impedance. If no neutral wire is connected, i.e. $Y_N = 0$ then the neutral voltage from (7.42) was much higher.

8. Magnetically Coupled Circuits

8.1 Mutual inductance

Calculate the phasor currents in the circuit of Fig. 8.1 if V_S = 12 V.

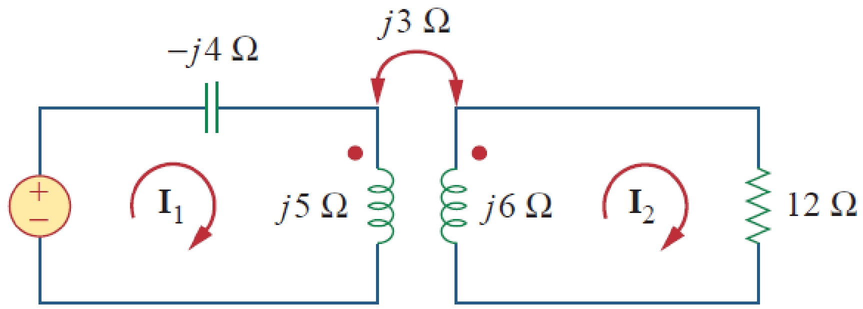

Figure 8.1 Circuit with mutual inductance

Solution

The first loop equation on the generator side applying KVL in the first loop is

$$-12 + (-j4 + j5)\boldsymbol{I}_1 - j3\boldsymbol{I}_2 \rightarrow j\boldsymbol{I}_1 - j3\boldsymbol{I}_2 = 12 \tag{8.1}$$

The second loop equation is

$$-j3\boldsymbol{I}_1 + (12 + j6)\boldsymbol{I}_2 = 0 \rightarrow \boldsymbol{I}_1 = \frac{(12 + j6)\boldsymbol{I}_2}{j3} = (2 - j4)\boldsymbol{I}_2 \tag{8.2}$$

Substituting (8.2) to (8.1)

$$(j2 + 4 - j3)\boldsymbol{I}_2 = (4 - j) = 12 \rightarrow \boldsymbol{I}_2 = \frac{12}{(4 - j)} = 2.91\ e^{j14.04°} \tag{8.3}$$

Finally, I_1 is derived from (8.2)

$$\boldsymbol{I}_1 = (2 - j4)\boldsymbol{I}_2 = (2 - j4)2.91\ e^{j14.04°} = 4.472\ e^{-j63.43°} \cdot 2.91\ e^{j14.04°} \tag{8.4}$$

thus,

$$\boldsymbol{I}_1 = 13.01\ e^{-j49.39°}\ A \tag{8.5}$$

8.2 Energy in a coupled circuit

Determine the coupling coefficient in the circuit shown in Fig. 8.2 and calculate the energy stored in the coupled inductors at *t = 1 s* if $v = 60\cos(4t + 30°)\ V$

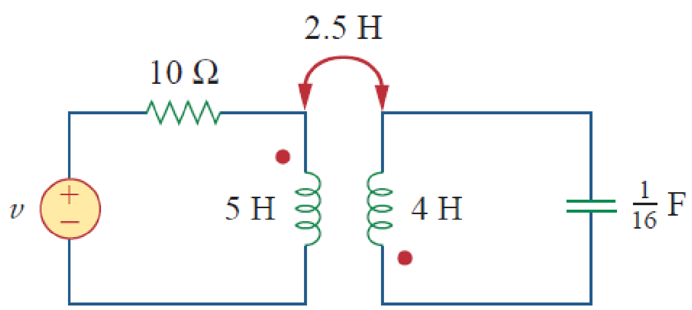

Figure 8.2 Magnetically coupled circuit

Solution

The coupling coefficient is defined as

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56 \tag{8.6}$$

The value of 0.56 shows a rather loose coupling between the coils. Far from k = 1.

The angular frequency of the source voltage is 4 rad/s, so the first loop equation in the phasor domain is

$$(10 + j20)\boldsymbol{I}_1 + j10\boldsymbol{I}_2 = 60\ e^{j30°} \tag{8.7}$$

The second loop equation in the phasor domain is

$$j10\boldsymbol{I}_1 + (j16 - j4)\boldsymbol{I}_2 = 0 \rightarrow \boldsymbol{I}_1 = -1.2\boldsymbol{I}_2 \tag{8.8}$$

Substituting (8.8) into (8.7) we have (8.9).

$$(-12 - j14)\boldsymbol{I}_2 = 60\ e^{j30°} \tag{8.9}$$

Thus,

$$\boldsymbol{I}_2 = 3.254\ e^{j160.6°}\ A \tag{8.10}$$

Now we can get $\boldsymbol{I_1}$ from (8.8)

$$\boldsymbol{I}_1 = -1.2\boldsymbol{I}_2 = 3.905\ e^{-j19.4°}\ A \tag{8.11}$$

Transforming the phasor currents in (8.10) and (8.11) to the time domain we get the primer and seconder currents as in (8.12) and (8.13).

$$\boldsymbol{I}_1 = 3.905\ e^{-j19.4°} \rightarrow i_1(t) = 3.905\cos(4t - 19.4°) \tag{8.12}$$

$$\boldsymbol{I}_2 = 3.254\ e^{j160.6°} \rightarrow i_2(t) = 3.254\cos(4t + 160.6°) \tag{8.13}$$

For the energy stored in the two inductors we need to know their currents at t =1 s. For the argument of the trigonometric part of (8.12) and (8.13) we get (8.14).

$$t = 1\ s \rightarrow 4t = 4\ rad = 229.2° \tag{8.14}$$

Thus, the instantaneous value of currents at t = 1 s are the following.

$$I_1 = 3.905\cos(229.2° - 19.4°) = -\ 3.389\ A \tag{8.15}$$

$$I_2 = 3.254\cos(229.2° + 160.6°) = 2.824\ A \tag{8.16}$$

The total energy in coupled inductors is obtained from (8.17).

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 \pm M I_1 I_2 \tag{8.17}$$

Substituting parameters, we get the total energy in joules as the following.

$$w = \frac{1}{2} \cdot 5 \cdot (-3.389)^2 + \frac{1}{2} \cdot 4 \cdot (2.824)^2 + 2.5 \cdot (-3.389) \cdot 2.824 = 20.73\, J$$

8.3 Linear transformers

Find the input impedance and current I_1 in the circuit shown in Fig. 8.3 when $\boldsymbol{V_S} = 50\, e^{j60°}\, V$, $\boldsymbol{Z_1} = 60 - j100\, \Omega$, $\boldsymbol{Z_2} = 30 + j40 \Omega$, $\boldsymbol{Z_L} = 80 + j60\, \Omega$.

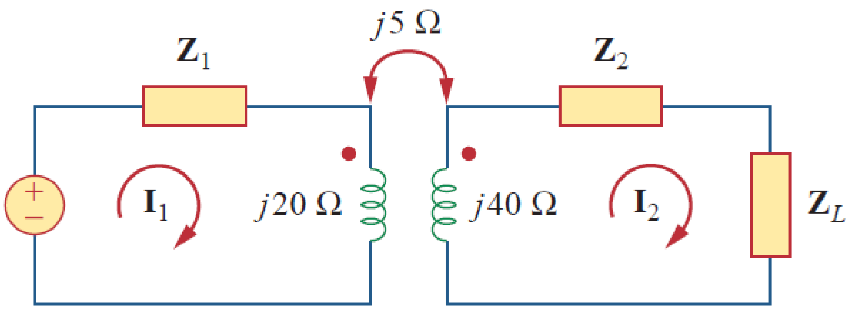

Figure 8.3 Linear transformer with source and loads.

Solution

The input impedance consists of two parts - the primary and the reflected (coupled) impedance part as shown in (8.18).

$$\boldsymbol{Z_{in}} = \{\boldsymbol{Z_1} + j\omega L_1\} + \left\{\frac{\omega^2 M^2}{\boldsymbol{Z_2} + j\omega L_2 + \boldsymbol{Z_L}}\right\} \tag{8.18}$$

Substituting the parameters into (8.18)

$$\boldsymbol{Z_{in}} = \boldsymbol{Z_1} + j20 + \frac{5^2}{j40 + \boldsymbol{Z_2} + \boldsymbol{Z_L}} = 60 - j100 + j20 + \frac{25}{110 + j140} \tag{8.19}$$

thus,

$$\boldsymbol{Z_{in}} = 60.09 - j80.11 = 100.14\, e^{-j53.1°}\, \Omega \tag{8.20}$$

The primary current is obtained in (8.21)

$$\boldsymbol{I_1} = \frac{\boldsymbol{V}}{\boldsymbol{Z_{in}}} = \frac{50\, e^{j60°}}{100.14\, e^{-j53.1°}} = 0.5\, e^{j113.1°}\, A \tag{8.21}$$

8.4 T-equivalent circuit of a linear transformer

Determine the T-equivalent circuit of the linear transformer shown in Fig. 8.4.

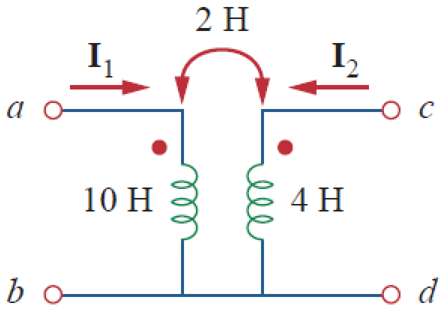

Figure 8.4 Linear transformer

Solution

To obtain the T-equivalent circuit of a linear transformer we use Fig. 8.5.

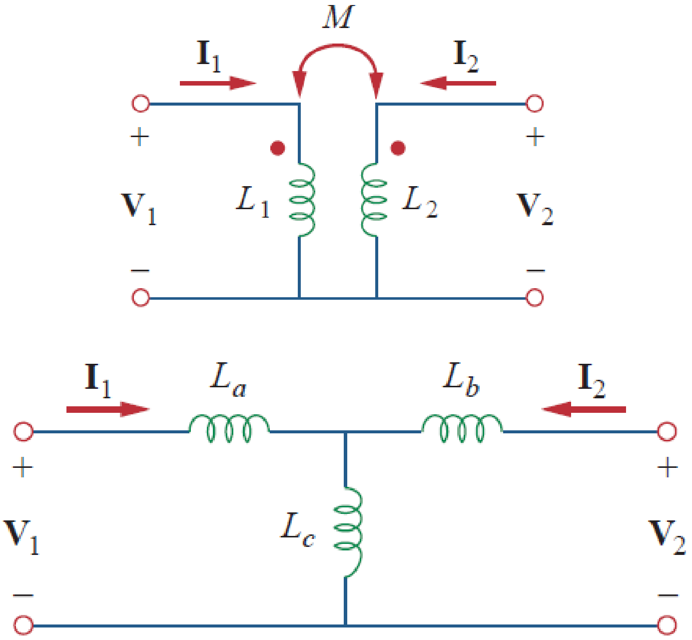

Figure 8.5 Linear transformer and its T equivalent circuit

The equivalence parameters of the T-circuit are given in (8.22

$$L_a = L_1 - M, \qquad L_b = L_2 - M\,, \qquad L_C = M \tag{8.22}$$

The self and mutual inductance parameters are given in Fig. 8.4.

$$L_1 = 10\,H\,, \qquad L_2 = 4\,H\,, \qquad M = 2\,H \tag{8.23}$$

Substituting the parameters from (8.23) into (8.22) we obtain

$$L_a = L_1 - M = 8\,H\,, \qquad L_b = L_2 - M = 2\,H\,, \qquad L_C = M = 2\,H \tag{8.24}$$

The T-equivalent circuit of Fig. 8.4 is given in Fig. 8.6.

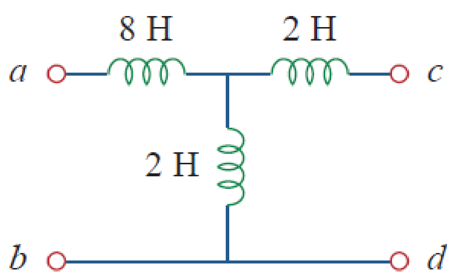

Figure 8.6 T-equivalent of Fig. 8.4

9. Frequency Response

9.1 Transfer function for RC circuits

For the *RC* circuit, shown in Fig. 9.1, obtain the transfer function as V_0/V_S voltage gain and its frequency response.

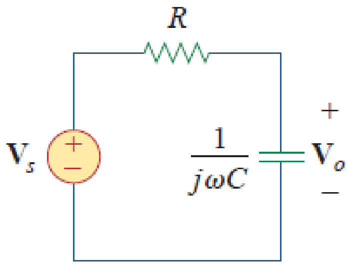

Figure 9.1 Low pass filter circuit

Solution

The transfer function as voltage gain is defined in (9.1).

$$\boldsymbol{H}(\omega) = \frac{\boldsymbol{V}_0(\omega)}{\boldsymbol{V}_S(\omega)} \tag{9.1}$$

Thus, the transfer function for the circuit in Fig 9.1 can be written applying voltage division.

$$\boldsymbol{H}(\omega) = \frac{{}^{1}/_{j\omega C}}{R + {}^{1}/_{j\omega C}} \tag{9.2}$$

Simplifying (9.2) and introducing the ω_0 break frequency parameter

$$\boldsymbol{H}(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\,{}^{\omega}/_{\omega_0}} \leftarrow \omega_0 = \frac{1}{RC} \tag{9.3}$$

The transfer function is a complex function. To examine its frequency response, we need to examine the amplitude and phase response separately, as written in (9.4) and (9.5).

$$H(\omega) = \frac{1}{\sqrt{1 + \left({}^{\omega}/_{\omega_0}\right)^2}} \tag{9.4}$$

$$\phi = -tan^{-1}\frac{\omega}{\omega_0} \tag{9.5}$$

The magnitude response, according to (9.4) is shown in Fig. 9.2 and the phase response of the transfer function, according to (9.5) is shown in Fig. 9.3.

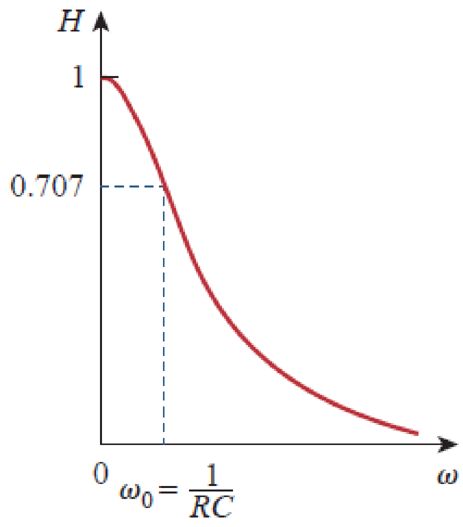

Figure 9.2 Low pass filter magnitude response

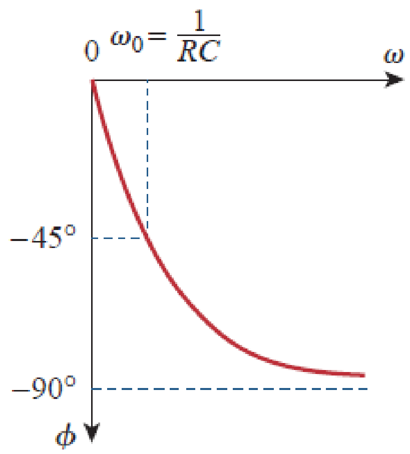

Figure 9.3 Low pass filter phase response

9.2 Transfer function for RL circuit

For the *RL* circuit, shown in Fig. 9.4, obtain the transfer function and its frequency response.

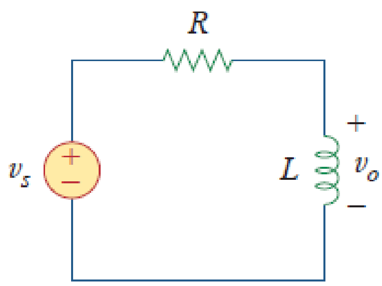

Figure 9.4 High pass filter circuit

Solution

The transfer function for the high pass filter circuit, shown in Fig. 9.4, is obtained using voltage division. The result is in (9.6).

$$\boldsymbol{H}(\omega) = \frac{j\omega L}{R + j\omega L} \tag{9.6}$$

Introducing the break frequency parameter, the transfer function becomes

$$\boldsymbol{H}(\omega) = \frac{j\,{}^{\omega}/_{\omega_0}}{1 + j\,{}^{\omega}/_{\omega_0}} \leftarrow \omega_0 = \frac{R}{L} \tag{9.7}$$

To draw the frequency response of the given RL circuit we have to draw the magnitude response and the phase response separately as we did previously in example 9.2. The results are in Fig. 9.5 and Fig. 9.6.

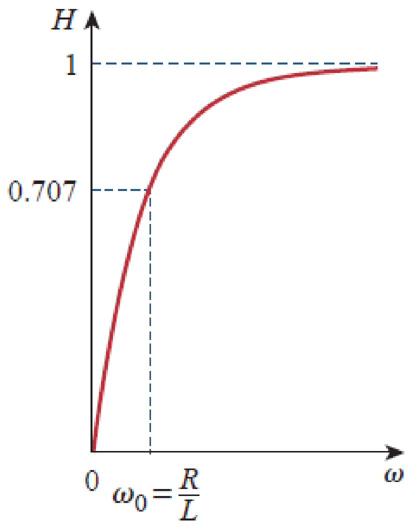

Figure 9.5 High pass filter magnitude response

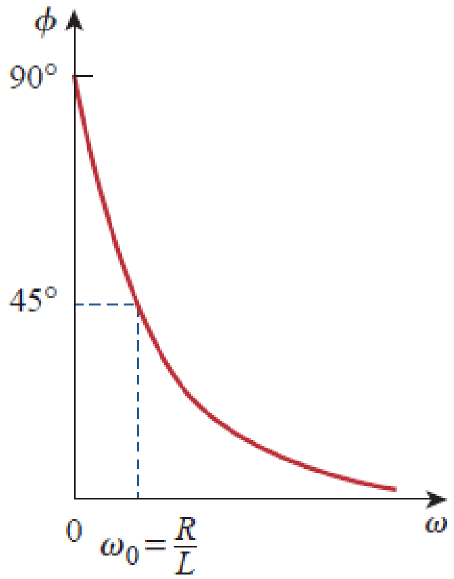

Figure 9.6 High pass filter phase response

9.3 Transfer impedance, zeros and poles

Calculate the transfer impedance of V_0/I_i and its poles and zeros for the circuit shown in Fig. 9.7.

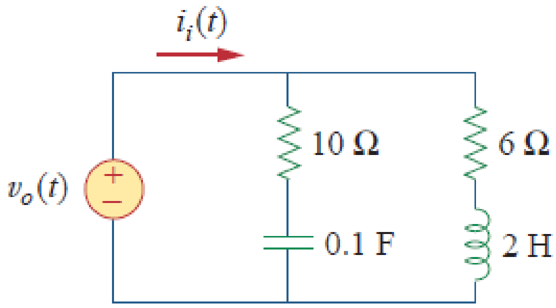

Figure 9.7 Circuit for transfer impedance calculation

Solution

The required transfer impedance, as a sort of transfer function, is obtained from (9.8). To make the equation short and simple we applied a note of s=jω.

$$\boldsymbol{Z}(\omega) = \frac{\boldsymbol{V}_0(\omega)}{\boldsymbol{I}_i(\omega)} = \left(10 + \frac{10}{s}\right) \times (6 + 2s) = (10 + 10s^{-1}) \times (6 + 2s) \tag{9.8}$$

Evaluating the equation, we obtain the following.

$$= \frac{60 + \ 60s^{-1} + 20s + 20}{16 + 10s^{-1} + 2s} = \frac{10(8s + 6 + 2s^2)}{2(8s + 5 + s^2)} = 10\frac{(s^2 + 4s + 3)}{s^2 + 8s + 5} \tag{9.9}$$

Zeros are the roots of the numerator of (9.9).

$$s_{1,2} = \frac{-4 \pm \sqrt{16 - 12}}{2} = \begin{cases} -1 \rightarrow z_1 = 1 \\ -3 \rightarrow z_2 = 3 \end{cases} \tag{9.10}$$

Poles are the roots of the denominator of (9.9).

$$s_{3,4} = \frac{-8 \pm \sqrt{64 - 20}}{2} = \begin{cases} -0.683 \rightarrow p_1 = 0.683 \\ -7.317 \rightarrow p_2 = 7.317 \end{cases} \tag{9.11}$$

Thus, the transfer impedance with applied zeros and poles is given in (9.12).

$$\boldsymbol{Z}(\omega) = \frac{\boldsymbol{V}_0(\omega)}{\boldsymbol{I}_i(\omega)} = \frac{10(s + 1)(s + 3)}{(s + 0.68)(s + 7.32)} \tag{9.12}$$

9.4 Bode plots for transfer function analysis

Construct the Bode plots for the transfer function given in (9.13).

$$\boldsymbol{H}(\omega) = \frac{j200\omega}{(j\omega + 2)(j\omega + 10)} \tag{9.13}$$

Solution

The first step, for drawing magnitude and phase response in Bode plots, is to convert the transfer function, given in (9.13) into 'standard form'. In the standard form we need to have only constant gain, zeros and poles at the origin, and simple and quadratic zeros and poles in the transfer function. The standard form of (9.13) is given in (9.14).

$$\boldsymbol{H}(\omega) = \frac{200}{2 \cdot 10} \cdot \frac{j\omega}{\left(1 + \frac{j\omega}{2}\right)\left(1 + \frac{j\omega}{10}\right)} \tag{9.14}$$

Thus, we have (9.15) for the magnitude plot and (9.16) for the phase plot from (9.14)..

$$H_{dB} = 20\log 10 + 20\log|j\omega| - 20\log\left|1+\frac{j\omega}{2}\right| - 20\log\left|1+\frac{j\omega}{10}\right| \tag{9.15}$$

$$\phi = 90° - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{\omega}{10} \tag{9.16}$$

The magnitude plot of (9.15) and phase plot of (9.16) are given in Fig. 9.8 and Fig. 9.9.

The construction steps can be easily followed by the given dashed lines. The final results, i.e. the sums of dashed lines, are given using continuous lines.

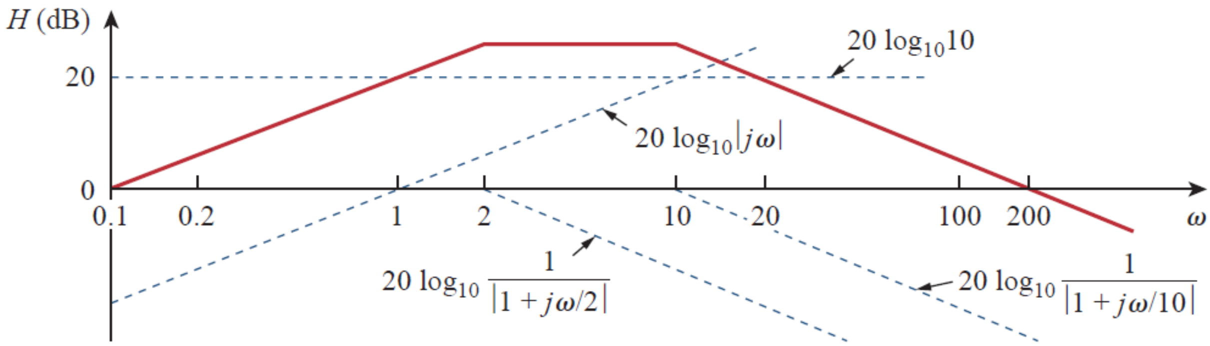

Figure 9.8 Magnitude response in Bode plot

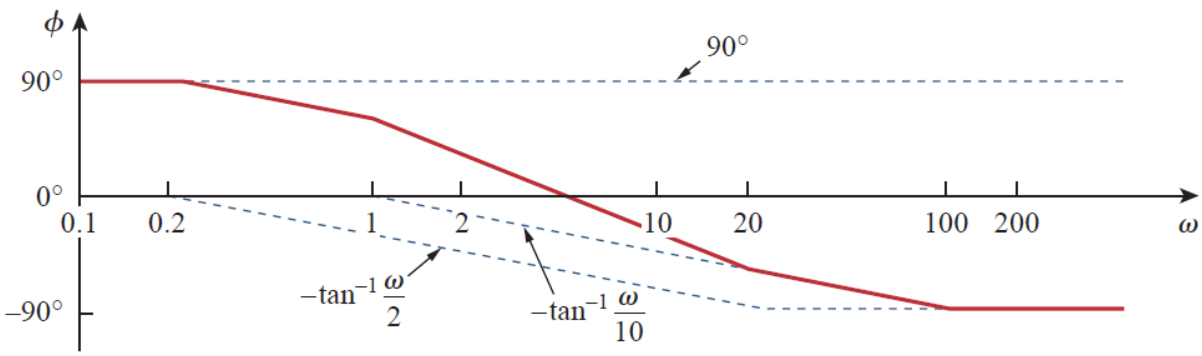

Figure 9.9 Phase response in Bode plot

9.5 Bode plots for transfer function synthesis

Find the transfer function for the Bode plot, given in Fig. 9.10.

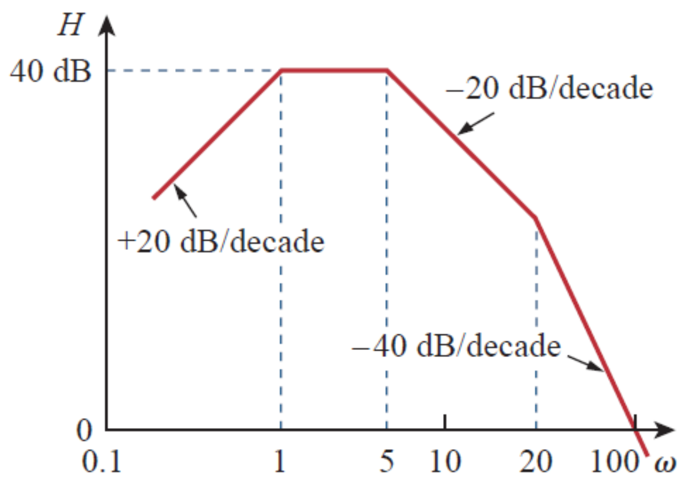

Figure 9.10 Bode plot example for transfer function synthesis

Solution

The given Bode plot is started with a +20 dB/decade asymptote at low frequency so the transfer function has zero at the origin.

$$\boldsymbol{H}(\omega) = \frac{j\omega(\dots)}{(\dots)} \tag{9.17}$$

The gain at ω = 1 rad/s is 40 dB thus a *K* constant is given by (9.18)

$$40\ dB = 20\log K \rightarrow K = 100 \rightarrow \boldsymbol{H}(\omega) = 100\frac{j\omega}{(\dots)} \tag{9.18}$$

ω = 1 rad/s is a corner break point where slope changes from +20 dB/decade to zero, so ω = 1 rad/s is pole as expressed in (9.19)

$$\boldsymbol{H}(\omega) = 100\frac{j\omega}{(1+j\omega)(\dots)} \tag{9.19}$$

The next break point is at ω = 5 rad/s where the slope changes from zero to –20 dB/decade, so ω = 5 rad/s is pole. Thus, we have (9.20)

$$\boldsymbol{H}(\omega) = 100\frac{j\omega}{(1+j\omega)(1+j\omega/5)(\dots)} \tag{9.20}$$

ω = 20 rad/s is a pole frequency because the slope changes from –20 dB/decade to –40 dB/decade. Hence,

$$\boldsymbol{H}(\omega) = 100\frac{j\omega}{(1+j\omega)(1+j\omega/5)(1+j\omega/20)} \tag{9.21}$$

After simplification of (9.21) we have a transfer function as in (9.22). The formula is given both in the frequency domain (as a function of *ω*) and in the Laplace domain (as a function of *s,* with *s = jω* substitution).

$$\boldsymbol{H}(\omega) = \frac{10^4 j\omega}{(j\omega+1)(j\omega+5)(j\omega+20)} \rightarrow \boldsymbol{H}(s) = \frac{10^4 s}{(s+1)(s+5)(s+20)} \tag{9.22}$$

10. Resonance Circuits

10.1 Series RLC circuit

Calculate the equivalent impedance, the circuit’s current, the power factor and draw the voltage phasor diagram in the circuit, shown in Fig. 10.1.

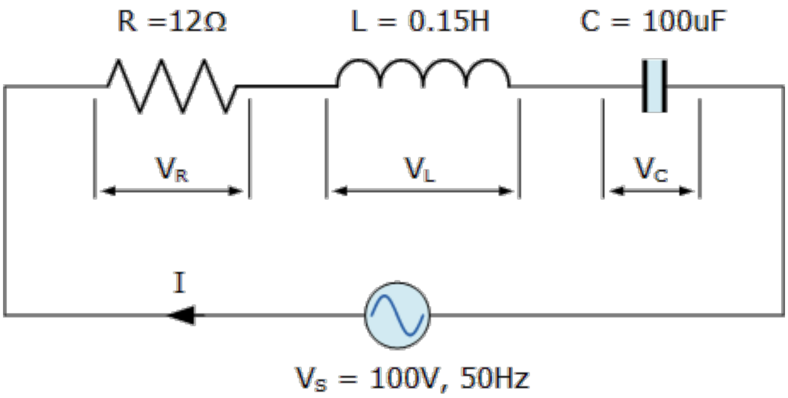

Figure 10.1 Series RLC circuit

Solution

For the impedance calculation we need to know the reactance of each reactive element.

$$X_L = \omega \cdot L = 2\pi \cdot 50 \cdot 0.15 = 47.13\ \Omega \tag{10.1}$$

$$X_C = \frac{1}{\omega \cdot C} = \frac{1}{2\pi \cdot 50 \cdot 100 \cdot 10^{-6}} = 31.83\ \Omega \tag{10.2}$$

Thus, the absolute value of circuit impedance is given in (10.3).

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = 19.4\ \Omega \tag{10.3}$$

Absolute value of circuit current is

$$I = \frac{V_S}{Z} = \frac{100}{19.4} = 5.15\ A \tag{10.4}$$

Thus, the voltage across the resistor is

$$V_R = I \cdot R = 5.15 \cdot 12 = 61.8\ V \tag{10.5}$$

and the voltages across the reactive elements are

$$V_L = I \cdot X_L = 5.15 \cdot 47.13 = 242.\ 4V \tag{10.6}$$

$$V_C = I \cdot X_C = 5.15 \cdot 31.83 = 163.5\ V \tag{10.7}$$

The power factor of the circuit is given in (10.9)

$$pf = \cos\varphi = \frac{R}{Z} = 0.619 \tag{10.9}$$

from which we obtain the phase angle between the source voltage and the current.

$$\varphi = 51.8 \tag{10.9}$$

Voltage across the inductor is higher than the voltage across the capacitor so the circuit is inductive at the given frequency. The phase angle demonstrates the 'lagging' phase i.e. the current is delayed to the voltage with the phase given in (10.9).

The phasor diagram of the examined circuit is given in Fig. 10.2.

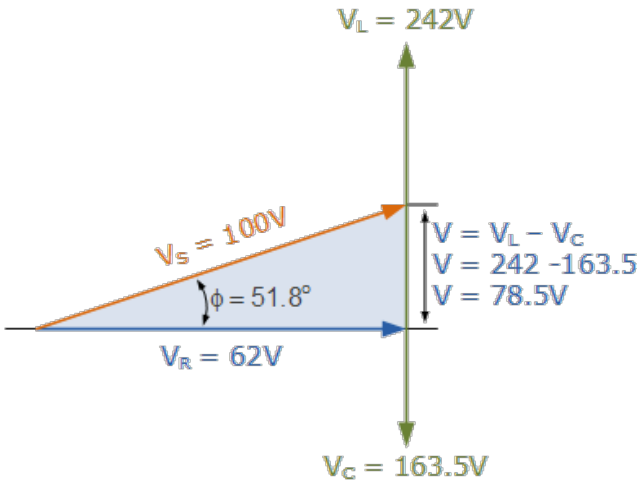

Figure 10.2 Phasor diagram for circuit in Fig. 10.1

10.2 Series resonance

The series RLC circuit is given in Fig. 10.3. Calculate the resonant frequency, the current at resonance, the voltage across the inductor and capacitor at resonance, the quality factor and the bandwidth of the circuit. Also sketch the corresponding current waveform for all frequencies.

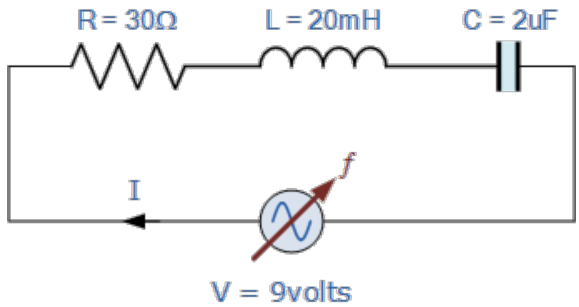

Figure 10.3 Series resonance circuit

Solution

The resonant frequency is given by the Thomson formula.

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.02 \cdot 2 \cdot 10^{-6}}} = 796\ Hz \tag{10.10}$$

Because the capacitor and inductor have the same impedance but the opposite sign, the equivalent impedance is simply the resistance *R,* at resonance. Thus, the current in the circuit is given by (10.11).

$$I = \frac{V}{R} = \frac{9}{30} = 300\ mA \tag{10.11}$$

The voltage across the resistor is 9 V, that is the source voltage because there was no 'common' voltage drop on the series connected LC elements at resonance. To obtain individual voltage across the inductor we need to know the inductive reactance.

$$X_L = \omega \cdot L = 2\pi \cdot 796 \cdot 0.02 = 100\,\Omega \tag{10.12}$$

The capacitive reactance has the same value at resonance, so there is no need to calculate the voltage separately across the capacitor.

$$V_L (= V_C) = I \cdot X_L = 0.3 \cdot 100 = 30\,V \tag{10.13}$$

The quality factor of the circuit is given by (10.14).

$$Q = \frac{X_L}{R} = \frac{100}{30} = 3.33 \tag{10.14}$$

Please note that the voltage across each of reactive elements in series resonance can be higher than the source voltage itself depending on the ratio of reactive and resistive impedances. The ratio of reactive voltage to the source voltage is the quality factor itself.

The bandwidth, according to its definition is given by (10.15).

$$\mathrm{BW} = \frac{f_r}{Q} = \frac{796}{3.33} = 238\,Hz \tag{10.15}$$

Thus, the lower and upper cut-off frequencies are the following.

$$f_L = f_r - \frac{1}{2}\mathrm{BW} = 796 - \frac{238}{2} = 677\,Hz \tag{10.16}$$

$$f_H = f_r + \frac{1}{2}\mathrm{BW} = 796 + \frac{238}{2} = 915\,Hz \tag{10.17}$$

The circuit current for all frequencies is shown in Fig. 10.4. The value of electric current at cut-off frequencies can be calculated as in (10.18) because the electric power is half its maximum value at cut-off frequency.

$$I_{cut-off} = \frac{I_r}{\sqrt{2}} = 0.707 \cdot 300 = 212\,mA \tag{10.18}$$

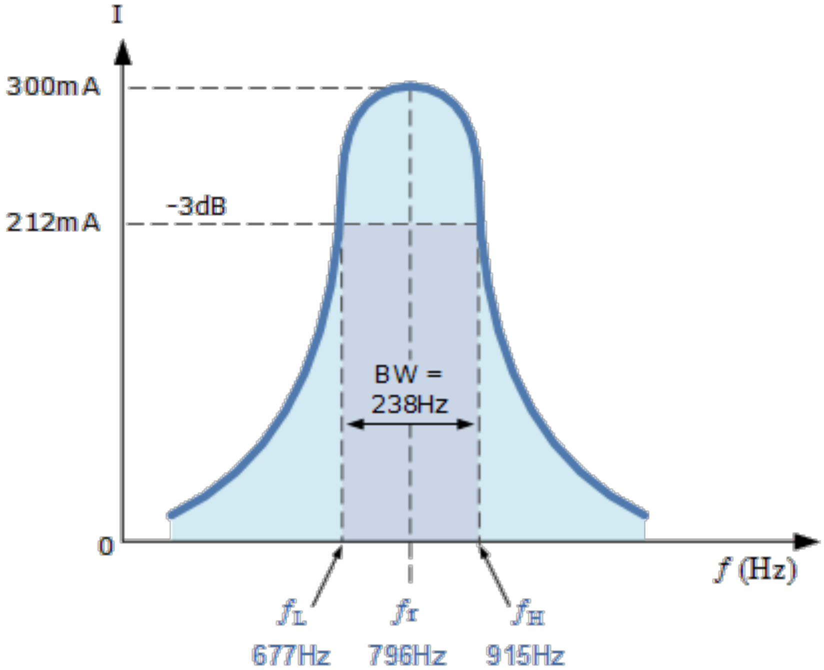

Figure 10.4 Circuit current vs. frequency in series resonance circuit

10.3 Parallel RLC circuits

Calculate the I_S, I_R, I_L, I_C, Z, and the phase angle for the circuit, shown in Fig. 10.5. Construct the current and admittance triangles representing the circuit.

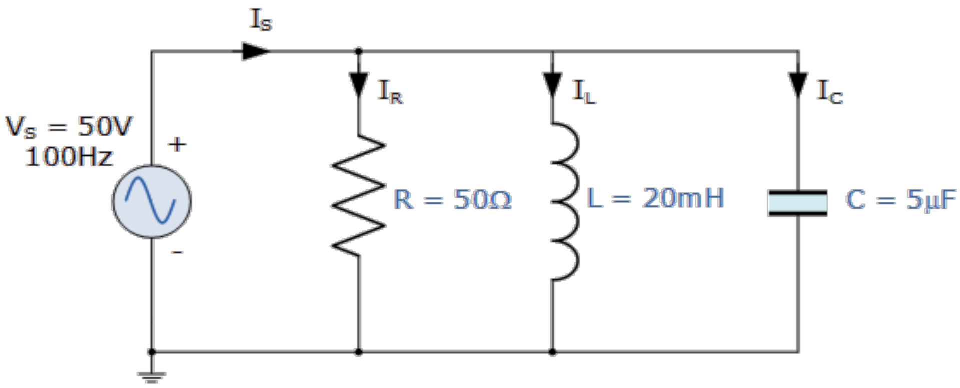

Figure 10.5 Parallel RLC circuit

Solution

According to the parallel connected circuit elements each branch has the same voltage, that is, V_S. To obtain branch currents we need to know the absolute value of impedance elements, i.e. the reactance of both reactive elements.

$$X_L = \omega \cdot L = 2\pi \cdot 100 \cdot 20 \cdot 10^{-3} = 12.6\ \Omega \tag{10.19}$$

$$X_C = \frac{1}{\omega \cdot C} = \frac{1}{2\pi \cdot 100 \cdot 5 \cdot 10^{-6}} = 318.3\ \Omega \tag{10.20}$$

Thus, the absolute value of circuit impedance is given by (10.21)

$$Z = \frac{1}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{X_L} - \frac{1}{X_C}\right)^2}} = \frac{1}{\sqrt{\left(\frac{1}{50}\right)^2 + \left(\frac{1}{12.6} - \frac{1}{318.3}\right)^2}} = \cdots = 12.7\,\Omega \tag{10.21}$$

The branch currents are the following

$$I_R = \frac{V_S}{R} = \frac{50}{50} = 1\,A \tag{10.22}$$

$$I_L = \frac{V_S}{X_L} = \frac{50}{12.6} = 3.9\,A \tag{10.23}$$

$$I_C = \frac{V_S}{X_C} = \frac{50}{318.3} = 160\,mA \tag{10.24}$$

and the current flows through the source is

$$I_S = \sqrt{I_R^2 + (I_L - I_C)^2} = \cdots = 3.87\,A \tag{10.25}$$

For the phase angle calculation, we use admittance parameters instead of impedances. The conversions are given in (10.26), (10.27), (10.28) and (10.29).

$$G = \frac{1}{R} = \frac{1}{50} = 20\,mS \tag{10.26}$$

$$B_L = \frac{1}{X_L} = \frac{1}{12.6} = 80\,mS \tag{10.27}$$

$$B_C = \frac{1}{X_C} = \frac{1}{318.3} = 3\,mS \tag{10.28}$$

$$Y = \frac{1}{Z} = \frac{1}{12.7} = 78\,mS \tag{10.29}$$

Thus, the phase angle between the source voltage and source current can be calculated as in (10.30).

$$\cos\varphi = \frac{G}{Y} = \frac{20}{78} = 0.256 \rightarrow \varphi = 75.3° \tag{10.30}$$

Current through the inductor is higher than current through the capacitor, so the circuit shows inductive behaviour at the given frequency. Circuit current is delayed to the source voltage by the phase angle, given in (10.30).

Current and admittance ‘triangles’, representing the circuit, are given in Fig. 10.6.

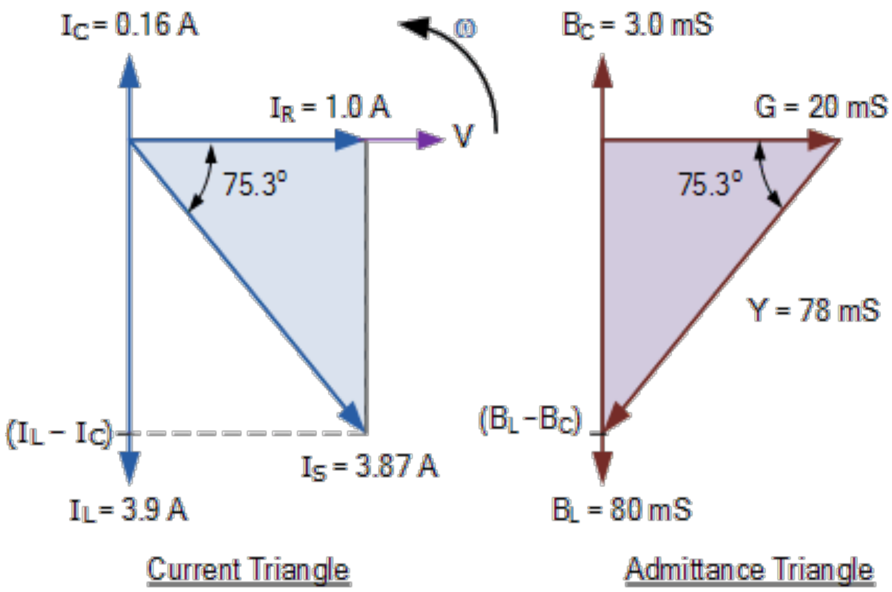

Figure 10.6 Current and admittance diagrams

10.4 Parallel resonance

Find f_r, Q, BW, the circuit current at resonance and current magnification for the circuit, shown in Fig. 10.7.

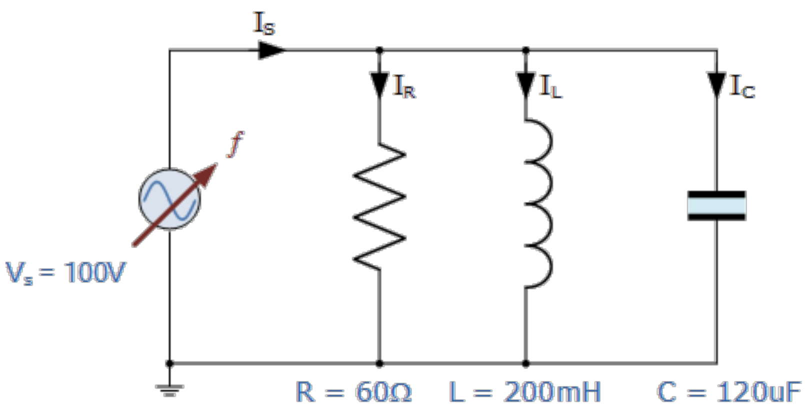

Figure 10.7 Parallel resonance circuit

Solution

The resonance frequency is obtained using Thomson's formula in (10.31).

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \cdot 120 \cdot 10^{-6}}} = 32.5\ Hz \tag{10.31}$$

For calculation of the quality factor we need to know the inductive (or capacitive) reactance at the resonance frequency.

$$X_L = \omega \cdot L = 2\pi \cdot 32.5 \cdot 0.2 = 40.8\ \Omega \tag{10.32}$$

Thus, the quality factor is

$$Q = \frac{R}{X_L} = \frac{60}{40.8} = 1.47 \tag{10.33}$$

and the bandwidth is

$$BW = \frac{f_r}{Q} = \frac{32.5}{1.47} = 22\ Hz \tag{10.34}$$

The lower and upper cut-off frequencies are given from resonance frequency and bandwidth, as calculated in (10.31) and (10.34)

$$f_L = f_r - \frac{1}{2}\mathrm{BW} = 32.5 - 11 = 21.5\ Hz \tag{10.35}$$

$$f_H = f_r + \frac{1}{2}\mathrm{BW} = 32.5 + 11 = 43.5\ Hz \tag{10.36}$$

Circuit current at resonance frequency is

$$I_S = \frac{V_S}{R} = \frac{100}{60} = 1.67\ A \tag{10.37}$$

Current magnification is calculated with resonance current and quality factor as in (10.38)

$$I_{MAG} = Q \cdot I_S = 1.47 \cdot 1.67 = 2.45\ A \tag{10.38}$$

Calculating the value of inductive (or capacitive) current at resonance, gives us the same value.

$$I_L = \frac{V_S}{X_L} = \frac{100}{40.8} = 2.45\ A \tag{10.39}$$

11. Multi-Wave Signals and Circuits

11.1 Fourier analysis for square wave signal

Determine the Fourier series of the waveform, shown in Fig. 11.1. Calculate the amplitude and phase spectra.

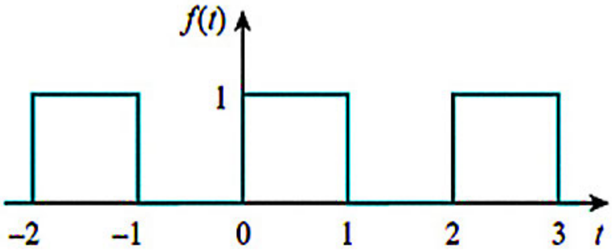

Figure 11.1 Square wave signal

Soution

The square waveform, given in Fig. 11.1, can be expressed in formula (11.1)

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \tag{11.1}$$

Period is 2 s, thus the fundamental angular frequency is

$$\omega_0 = \frac{2\pi}{T} = \pi \tag{11.2}$$

Using the Fourier analysis, we determine A_0, A_k and B_k coefficients. The DC component (A0) is given in (11.3)

$$A_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\left(\int_0^1 1dt + \int_1^2 0dt\right) = \frac{1}{2} \tag{11.3}$$

Amplitude of cosine components in the Fourier series are

$$A_k = \frac{2}{T}\int_0^T f(t)\cos k\omega_0 t\, dt = \frac{2}{2}\left(\int_0^1 1\cos k\pi t\, dt + \int_1^2 0\cos k\pi t\, dt\right) \tag{11.4}$$

Evaluating (11.4) we find no cosine components in the Fourier series.

$$A_k = \frac{1}{k\pi}\sin k\pi = 0 \tag{11.5}$$

Amplitude of sinusoidal components in the Fourier series are

$$B_k = \frac{2}{T}\int_0^T f(t)\sin k\omega_0 t\, dt = \frac{2}{2}\left(\int_0^1 1\sin k\pi t\, dt + \int_1^2 0\sin k\pi t\, dt\right) \tag{11.6}$$

Evaluating (11.6) we can write (11.7).

$$B_k = \left.\frac{-\cos k\pi t}{k\pi}\right|_0^1 = \frac{1}{k\pi}(-\cos k\pi + 1) \tag{11.7}$$

Because $\cos k\pi$ gives-1 for odd and 1 for even numbers we can write it as in (11.8).

$$B_k = \begin{cases} 2/k\pi & k = odd \\ 0 & k = even \end{cases} \tag{11.8}$$

Thus, the Fourier series by the coefficients given in (11.3), (11.5) and (11.8)

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sin \pi t + \frac{2}{3\pi} \sin 3\pi t + \frac{2}{5\pi} \sin 5\pi t + \frac{2}{7\pi} \sin 7\pi t + \cdots \tag{11.9}$$

or, written in short

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{i=1}^{\infty} \frac{1}{n} \sin n\pi t, \qquad n = 2i - 1 \tag{11.10}$$

To obtain the spectra of the square waveform we must find C_k and ϕ_k parameters. As no A_k coefficients exist in the Fourier series, C_k simply equals to B_k and $\phi_k = 0$.

11.2 Fourier analysis for saw wave signal

Obtain the Fourier series for the periodic function in Figure 11.2 and plot the amplitude and phase spectra.

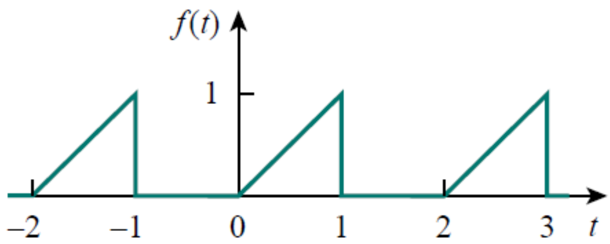

Figure 11.2 Saw wave signal

Solution

The *f(t)* function, represented in Fig. 11.2 is given in (11.11)

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases} \tag{11.11}$$

Because the period is 2 s (see Fig. 11.2), the fundamental angular frequency is

$$\omega_0 = \frac{2\pi}{T} = \pi \tag{11.12}$$

Determining the Fourier coefficient A_0, that is, the average value of *f(t)* we obtain the following.

$$A_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2} \left(\int_0^1 t\, dt + \int_1^2 0 dt \right) = \frac{1}{2} \left. \frac{t^2}{2} \right|_0^1 = \frac{1}{4} \tag{11.13}$$

A_k coefficients can be obtained from (11.14).

$$A_k = \frac{2}{T} \int_0^T f(t) \cos k\omega_0 t\, dt = \frac{2}{2} \left(\int_0^1 \mathrm{t} \cos k\pi t\, dt + 0 \right) \tag{11.14}$$

Thus, the result is the following.

$$A_k = \left[\frac{1}{k^2\pi^2}\cos k\pi t + \frac{1}{k\pi}\sin k\pi t\right]\Bigg|_0^1 = \frac{1}{k^2\pi^2}(\cos k\pi - 1) = \frac{(-1)^k - 1}{k^2\pi^2} \tag{11.15}$$

B_k coefficients can be obtained from (11.16).

$$B_k = \frac{2}{T}\int_0^T f(t)\, sin\, k\omega_0 t\, dt = \frac{2}{2}\left(\int_0^1 \mathrm{t}\, sin\, k\pi t\, dt + \int_1^2 0\, sin\, k\pi t\, dt\right) \tag{11.16}$$

Thus, the result is the following.

$$B_k = \left[\frac{1}{k^2\pi^2}\sin k\pi t - \frac{t}{k\pi}\cos k\pi t\right]\Bigg|_0^1 = 0 - \frac{\cos k\pi}{k\pi} = \frac{(-1)^{k+1}}{k\pi} \tag{11.17}$$

By substituting the calculated coefficients (11.13), (11.15), (11.17) into the formula of the Fourier series, gives us the result in (11.18)

$$f(t) = \frac{1}{4} + \sum_{i=1}^{\infty}\left[\frac{(-1)^k - 1}{(k\pi)^2}\cos k\pi t + \frac{(-1)^{k+1}}{k\pi} sin\, k\pi t\right] \tag{11.18}$$

Finally, we must find the C_k and φ_k parameters for plotting the amplitude and phase spectra. For even components

$$A_k = 0, B_k = -\frac{1}{k\pi} \leftarrow k = 2, 4, \ldots \tag{11.19}$$

thus,

$$\boldsymbol{C}_k = A_k - jB_k = 0 + j\frac{1}{k\pi} = \frac{1}{k\pi}e^{j90^\circ} \leftarrow k = 2, 4, \ldots \tag{11.20}$$

For odd components

$$A_k = -\frac{2}{(k\pi)^2}, B_k = \frac{1}{k\pi} \leftarrow k = 1, 3, \ldots \tag{11.21}$$

Thus,

$$\boldsymbol{C}_k = A_k - jB_k = -\frac{2}{(k\pi)^2} - j\frac{1}{k\pi} = C_k e^{j\varphi_k} \leftarrow k = 1, 3, \ldots \tag{11.22}$$

Transforming algebraic form to polar form, we obtain

$$\left.\begin{aligned} C_k = \sqrt{A_k^2 + B_k^2} = \sqrt{\frac{4}{(k\pi)^4} + \frac{1}{(k\pi)^2}} = \frac{\sqrt{4 + (k\pi)^2}}{(k\pi)^2} \\ \varphi_k = 180^\circ + tan^{-1}\frac{k\pi}{2} \end{aligned}\right\} \leftarrow k = 1, 3, \ldots \tag{11.23}$$

Calculating amplitude and phase values from (11.20) and (11.23) we can plot the spectra as shown in Fig. 11.3 and Fig. 11.4.

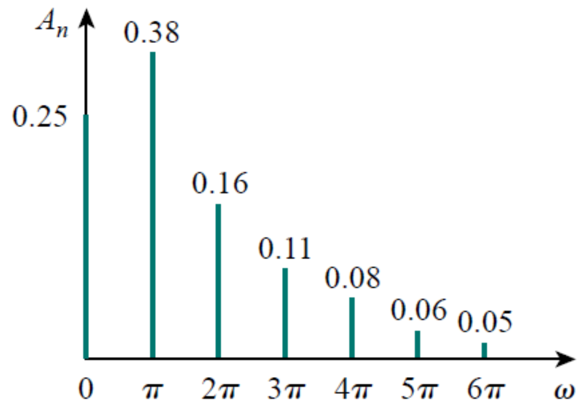

Figure 11.3 Amplitude spectra of the waveform in Fig. 11.2

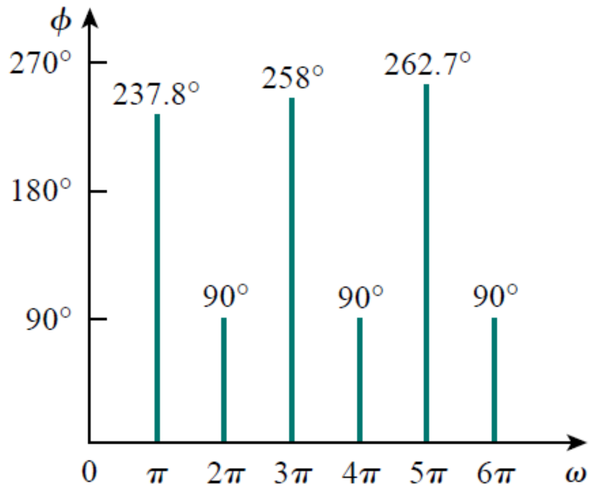

Figure 11.4 Phase spectra of the waveform in Fig. 11.2

11.3 Circuit analysis with square wave excitation

Find the $v_O(t)$ response voltage across the inductor for the circuit shown in Fig. 11.5 if the source voltage is given as in (11.24), that is, the Fourier series of square wave signal.

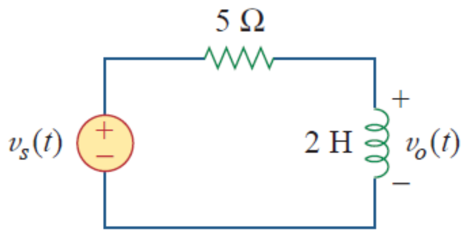

Figure 11.5 Circuit with square-wave source

$$v_S(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n} sin\, n\pi t\,, \qquad n = 2k - 1 \tag{11.24}$$

Solution

For the n-th harmonic (that is sinusoidal time varying function) we can use the method of circuit analysis in the phasor domain i.e. we simply apply the voltage division formula

$$\boldsymbol{V}_{On} = \boldsymbol{V}_{Sn}\frac{j\omega_n L}{R + j\omega_n L} = \boldsymbol{V}_{Sn}\frac{j2n\pi}{5 + j2n\pi} \tag{11.25}$$

$$\omega_n = n \cdot \omega_1 = n \cdot \pi \tag{11.26}$$

Substituting each phasor component separately from (11.24) into (11.25) we obtain each phasor component of voltage across the inductor.

For n = 0, i.e. for the DC component of the excitation

$$V_{S0} = \frac{1}{2} \to V_{O0} = 0 \tag{11.27}$$

For the calculation of the n-th harmonic of the output voltage we transform sinusoidal components in (11.24) to cosine for convenience. The result is given in (11.28).

$$v_S(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}cos(n\pi t - 90°)\,, \qquad n = 2k - 1 \tag{11.28}$$

The n-th component in (11.28) in the phasor domain is obtained in (11.29)

$$\boldsymbol{V}_{Sn} = \frac{2}{n\pi}e^{-j90°} = \frac{-j2}{n\pi} \tag{11.29}$$

The n-th harmonic of the output voltage from (11.25) and (11.29) is

$$\boldsymbol{V}_{On} = \frac{-j2}{n\pi}\frac{j2n\pi}{5 + j2n\pi} = \frac{4e^{-jtan^{-1}\frac{2n\pi}{5}}}{\sqrt{25 + 4n^2\pi^2}} \tag{11.30}$$

Transforming the results in (11.27) and (11.30) into the time domain gives us the output voltage as a sum of the harmonic components.

$$v_0(t) = 0 + \sum_{k=1}^{\infty}\frac{4}{\sqrt{25 + 4n^2\pi^2}}cos\left(n\pi t - tan^{-1}\frac{2n\pi}{5}\right), \qquad n = 2k - 1 \tag{11.31}$$

Thus,

$$v_0(t) = 0.50\,cos(\pi t - 51.5°) + 0.2\,cos(3\pi t - 75°) + 0.13\,cos(5\pi t - 81°) + \cdots \tag{11.32}$$

11.4 Average power in multi-wave circuits

Determine the average power supplied to the circuit in Fig. 11.6 if the current source provides a time varying signal, approximated as in (11.33)

Note: The Fourier series of the multi-wave current is approximated with its first 3 components for convenient calculation only. The approximation of a definite (few) number of components is often used in practice and can be justified by the convergence criteria of the Fourier series because its higher number component has a lower amplitude.

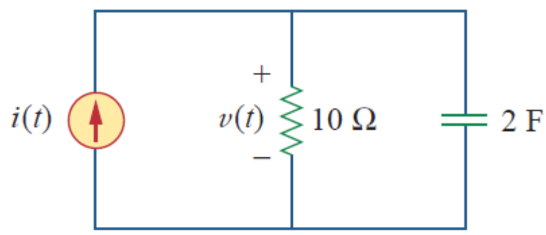

Figure 11.6 Circuit example for average power calculation

$$i(t) = 2 + 10\,cos(t + 10°) + 6\,cos(3t + 45°)\ A \tag{11.33}$$

Solution

For average power we know the current, given in (11.32) and the multi-wave voltage has to be determined. For this calculation the circuit impedance at a certain frequency is

$$\boldsymbol{Z} = 10 \times \frac{1}{j2\omega} = \frac{10}{1 + j20\omega} \tag{11.34}$$

and the voltage is

$$\boldsymbol{V} = \boldsymbol{I}\,\boldsymbol{Z} = \frac{10\,\boldsymbol{I}}{1 + j20\omega} = \frac{10\,\boldsymbol{I}}{\sqrt{1 + 400\omega^2}\ e^{jtan^{-1}20\omega}} \tag{11.35}$$

We have three components if the multi-wave current in its Fourier series and voltage components can be obtained as the following.

$$\omega = 0 \rightarrow \boldsymbol{I}_0 = 2\ A \rightarrow \boldsymbol{V} = 2 \cdot 10 = 20\ V \tag{11.36}$$

$$\omega = 1 \rightarrow \boldsymbol{I}_1 = 10\ e^{j10°} \rightarrow \boldsymbol{V} = \frac{10 \cdot 10\ e^{j10°}}{\sqrt{1 + 400}\ e^{jtan^{-1}20}} = 5\ e^{-j77.14°}\ V \tag{11.37}$$

$$\omega = 3 \rightarrow \boldsymbol{I}_3 = 6\ e^{j45°} \rightarrow \boldsymbol{V} = \frac{10 \cdot 6\ e^{j45°}}{\sqrt{1 + 3600}\ e^{jtan^{-1}60}} = 1\ e^{-j44.05°}\ V \tag{11.38}$$

Voltage v(t) is the outcome of (11.36), (11.37) and (11.38).

$$v(t) = 20 + 5\,cos(t - 77.14°) + 1\,cos(3t - 44.05°)\ V \tag{11.39}$$

The total average power is the sum of the average power of each component.

$$P = V_0 I_0 + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n\,cos(\varphi_{Un} - \varphi_{In}) \tag{11.40}$$

By substituting the appropriate current and voltage components into (11.34)

$$P = 20 \cdot 2 + \frac{5 \cdot 10}{2} cos(-77.14° - 10°) + \frac{1 \cdot 6}{2}\ cos(-44.05° - 45°) \tag{11.41}$$

$$P = 40 + 1.247 + 0.05 = 41.3\ W \tag{11.42}$$

An alternative way for average power calculation is through knowing that the average power is dissipated on the resistor only. Knowing the resistors' voltage for each component gives us (11.43).

$$P = \frac{V_0^2}{R} + \frac{1}{2}\sum_i \frac{V_i^2}{R} = \frac{400}{10} + \frac{1}{2}\frac{25}{10} + \frac{1}{2}\frac{1}{10} = 40 + 1.25 + 0.05 = 41.3\ W \tag{11.43}$$

Of course, the result is the same as in (11.42)

11.5 RMS value of a multi-wave signal

Estimate the RMS value of the voltage given by the Fourier series expansion in (11.44)

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2} (\cos nt - n \sin nt) \tag{11.44}$$

Solution

The Fourier coefficients in function (11.44) are the following

$$A_0 = 1,\ A_k = \frac{2(-1)^n}{1+n^2},\ B_k = \frac{2n(-1)^n}{1+n^2} \tag{11.45}$$

For the RMS value we need C_k and φ_k.

$$C_k = \sqrt{A_k^2 + B_k^2} = \frac{2(-1)^n}{\sqrt{1+n^2}},\ \varphi_k = tan^{-1} \frac{B_k}{A_k} = tan^{-1} n \tag{11.46}$$

Thus,

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + tan^{-1} n) \tag{11.47}$$

$$\begin{gathered} v(t) = 1 - 1.41 \cos(t + 45°) + 0.89 \cos(2t + 63.45°) \\ -0.63 \cos(3t + 71.56°) - 0.48 \cos(4t + 78.7°) + \cdots \end{gathered} \tag{11.48}$$

The RMS value of the multi-wave signal, given in (11.45) is

$$\begin{gathered} V_{rms} = \sqrt{A_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} C_k^2} \\ \approx \sqrt{1^2 + \frac{1}{2}[(-1.41)^2 + (0.89)^2 + (-0.63)^2 + (-0.48)^2 + \cdots]} \\ = \sqrt{2.72} = 1.65\ V \end{gathered} \tag{11.49}$$

Comment

Without giving details, the original function for the Fourier analysis was the function described in (11.50)

$$v(t) = \frac{\pi e^t}{\sinh \pi}, \qquad -\pi < t < \pi\ , \qquad v(t(= v(t+T) \tag{11.50}$$

The RMS value of this time varying function is $V_{RMS} = 1.776\ V$. We approximated the function with its first five components only, so the estimation in (11.49) is good enough.

12. Two-Port Networks

12.1 Impedance or open circuit parameters

Determine the impedance parameters of the T-section circuit shown in Fig. 12.1.

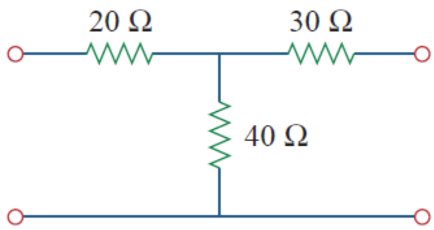

Figure 12.1 T-section circuit

Solution

The impedance characteristic of a two-port network is described by (12.1) and (12.2) equations.

$$\boldsymbol{V}_1 = \boldsymbol{z}_{11}\boldsymbol{I}_1 + \boldsymbol{z}_{12}\boldsymbol{I}_2 \tag{12.1}$$

$$\boldsymbol{V}_2 = \boldsymbol{z}_{21}\boldsymbol{I}_1 + \boldsymbol{z}_{22}\boldsymbol{I}_2 \tag{12.2}$$

From these characteristic equations, the open circuit input impedance (z_{11}) and open circuit transfer impedance (z_{21}) can be determined using the circuit in Fig. 12.2.

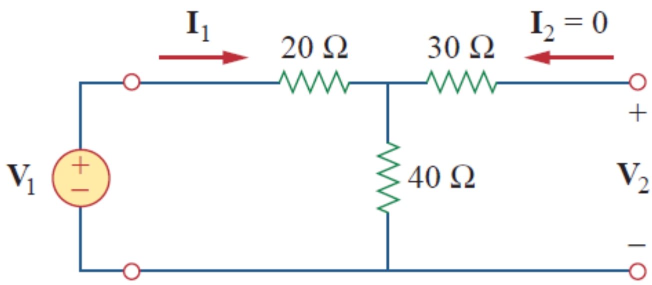

Figure 12.2 Open output condition

Thus,

$$\boldsymbol{z}_{11} = \left.\frac{\boldsymbol{V}_1}{\boldsymbol{I}_1}\right|_{\boldsymbol{I}_2=0} = \frac{\boldsymbol{I}_1 \cdot (20 + 40)}{\boldsymbol{I}_1} = 60\ \Omega \tag{12.3}$$

$$\boldsymbol{z}_{21} = \left.\frac{\boldsymbol{V}_2}{\boldsymbol{I}_1}\right|_{\boldsymbol{I}_2=0} = \frac{\boldsymbol{I}_1 \cdot 40}{\boldsymbol{I}_1} = 40\Omega \tag{12.4}$$

From (12.1) and (12.2) the characteristic equations for open circuit output impedance (z_{22}) and open circuit transfer impedance (z_{12}) can be determined using the circuit in Fig. 12.3.

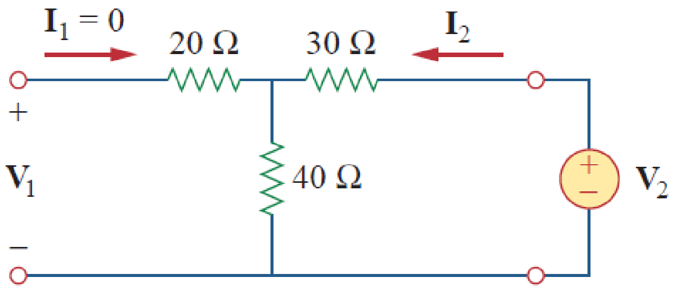

Figure 12.3 Open input condition

Applying the open circuit conditions, gives the following results.

$$\boldsymbol{z}_{12} = \left.\frac{\boldsymbol{V}_1}{\boldsymbol{I}_2}\right|_{\boldsymbol{I}_1=0} = \frac{\boldsymbol{I}_2 \cdot 40}{\boldsymbol{I}_2} = 40\Omega \tag{12.5}$$

$$\boldsymbol{z}_{22} = \left.\frac{\boldsymbol{V}_2}{\boldsymbol{I}_2}\right|_{\boldsymbol{I}_1=0} = \frac{\boldsymbol{I}_2 \cdot (30+40)}{\boldsymbol{I}_2} = 70\Omega \tag{12.6}$$

Thus, the impedance matrix for the T-section circuit in Fig. 12.1 is

$$\boldsymbol{Z} = \begin{bmatrix} 60 & 40 \\ 40 & 70 \end{bmatrix} \Omega \tag{12.7}$$

Because the transfer impedances are equal to each other, i.e. $\boldsymbol{z}_{12} = \boldsymbol{z}_{21}$, the circuit is reciprocal.

12.2 Hybrid parameters

Determine the hybrid parameters for the T-section circuit, shown in Fig. 12.4.

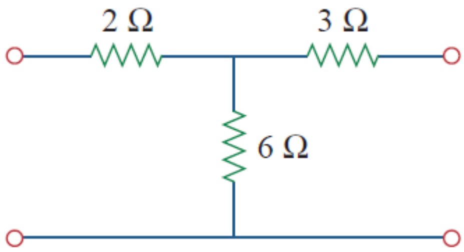

Figure 12.4 T-section circuit

Solution

The hybrid characteristic equations are given in (12.8) and (12.9).

$$\boldsymbol{V}_1 = \boldsymbol{h}_{11}\boldsymbol{I}_1 + \boldsymbol{h}_{12}\boldsymbol{V}_2 \tag{12.8}$$

$$\boldsymbol{I}_2 = \boldsymbol{h}_{21}\boldsymbol{I}_1 + \boldsymbol{h}_{22}\boldsymbol{V}_2 \tag{12.9}$$

Applying the short circuit condition, shown in Fig. 12.5, h_{11} parameter can be determined from (12.8).

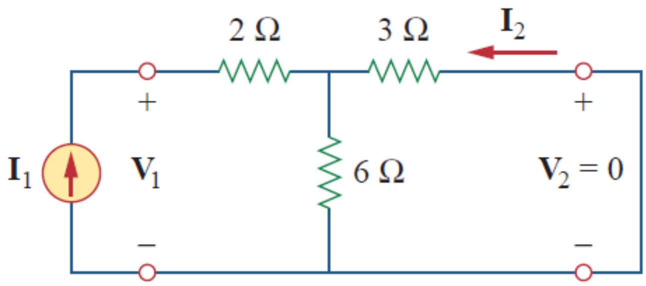

Figure 12.5 Short circuit output condition

From (12.8) we can write

$$\boldsymbol{V}_1 = \boldsymbol{h}_{11}\boldsymbol{I}_1 + \boldsymbol{h}_{12}\, 0 \tag{12.10}$$

thus,

$$\boldsymbol{h}_{11} = \left.\frac{\boldsymbol{V}_1}{\boldsymbol{I}_1}\right|_{\boldsymbol{V}_2=0} = (2+3) \times 6 = 4\,\Omega \tag{12.11}$$

To find h_{21} we can write the current division for the circuit in Fig. 12.5.

$$-\boldsymbol{I}_2 = \frac{6}{6+3}\boldsymbol{I}_1 = \frac{2}{3}\boldsymbol{I}_1 \tag{12.12}$$

Thus,

$$\boldsymbol{h}_{21} = \left.\frac{\boldsymbol{I}_2}{\boldsymbol{I}_1}\right|_{\boldsymbol{V}_2=0} = -\frac{2}{3} \tag{12.13}$$

Now we calculate h_{12} by applying the open input condition, using the circuit in Fig. 12.6.

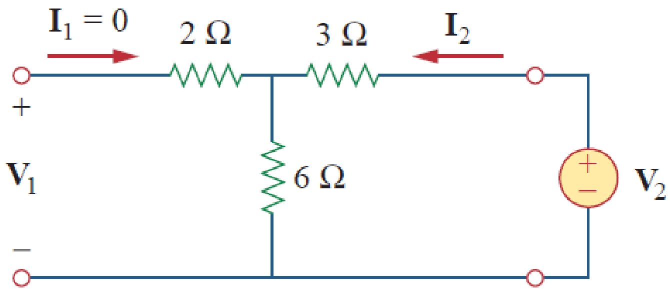

Figure 12.6 Open input condition

Using voltage division we can write the following

$$\boldsymbol{V}_1 = \frac{6}{6+3}\boldsymbol{V}_2 = \frac{2}{3}\boldsymbol{V}_2 \tag{12.14}$$

Substituting 12.14 into 12.8 gives us the h_{12} parameter

$$\boldsymbol{h}_{12} = \left.\frac{\boldsymbol{V}_1}{\boldsymbol{V}_2}\right|_{\boldsymbol{I}_1=0} = \frac{2}{3} \tag{12.15}$$

Applying KVL for the loop in Fig. 12.6

$$\boldsymbol{V}_2 = (3+6)\,\boldsymbol{I}_2 = 9\boldsymbol{I}_2 \tag{12.16}$$

thus,

$$\boldsymbol{h}_{22} = \left.\frac{\boldsymbol{I}_2}{\boldsymbol{V}_2}\right|_{\boldsymbol{I}_1=0} = \frac{1}{9}\ S \tag{12.17}$$

In the hybrid matrix some parameters are measured in siemens, some of them in ohms and others have no dimension, according to whether they are *mixed* or *hybrid* parameter sets.

12.3 Two-port network analysis

Find I_1 and I_2 in the circuit in Fig. 12.7.

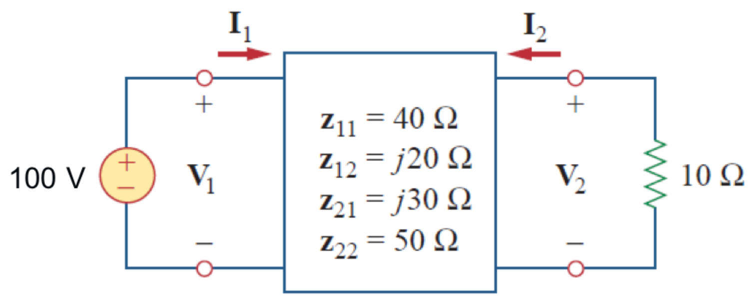

Figure 12.6 Network with z parameters

Solution

First of all, we identify the circuit in non-reciprocal form because transfer impedances are not equal to each other as z_{12} = j20 Ω and z_{21} = j30 Ω.

The impedance characteristic equations are the following.

$$\boldsymbol{V}_1 = 40\boldsymbol{I}_1 + j20\boldsymbol{I}_2 \tag{12.18}$$

$$\boldsymbol{V}_2 = j30\boldsymbol{I}_1 + 50\boldsymbol{I}_2 \tag{12.19}$$

Applying the input and output conditions as seen in Fig. 12.6, we can say

$$\boldsymbol{V}_1 = 100\,, \boldsymbol{V}_2 = -10\boldsymbol{I}_2 \tag{12.20}$$

And substituting (12.20) conditions into (12.18) and (12.19)

$$100 = 40\boldsymbol{I}_1 + j20\boldsymbol{I}_2 \tag{12.21}$$

$$-10\boldsymbol{I}_2 = j30\boldsymbol{I}_1 + 50\boldsymbol{I}_2 \rightarrow \boldsymbol{I}_1 = j2\boldsymbol{I}_2 \tag{12.22}$$

Because I_1 is derived from (12.22), we substitute this to (12.21) and I_2 can also be expressed.

$$100 = j80\boldsymbol{I}_2 + j20\boldsymbol{I}_2 \rightarrow \boldsymbol{I}_2 = -\mathrm{j}\ \mathrm{A} \tag{12.23}$$

Finally, the value of I_1 from (12.22) is

$$\boldsymbol{I}_1 = j2\boldsymbol{I}_2 = 2\ A \tag{12.24}$$

12.4 Admittance or short circuit parameters

Determine the admittance parameters for the two-ports shown in Fig. 12.7.

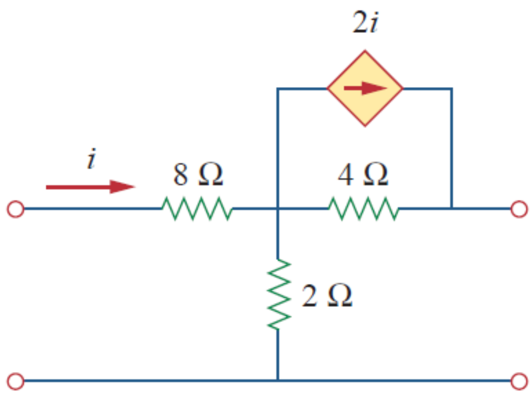

Figure 12.7 Circuit with dependent source

Solution

The admittance characteristic equations are given in (12.25) and (12.26).

$$\boldsymbol{I}_1 = \boldsymbol{y}_{11}\boldsymbol{V}_1 + \boldsymbol{y}_{12}\boldsymbol{V}_2 \tag{12.25}$$

$$\boldsymbol{I}_2 = \boldsymbol{y}_{21}\boldsymbol{V}_1 + \boldsymbol{y}_{22}\boldsymbol{V}_2 \tag{12.26}$$

Input admittance from (12.25) with the applied short circuit condition at output is

$$\boldsymbol{y}_{11} = \frac{\boldsymbol{I}_1}{\boldsymbol{V}_1}\bigg|_{\boldsymbol{V}_2=0} \tag{12.27}$$

The circuit we examine for determining y_{11} is shown in Fig. 12.8.

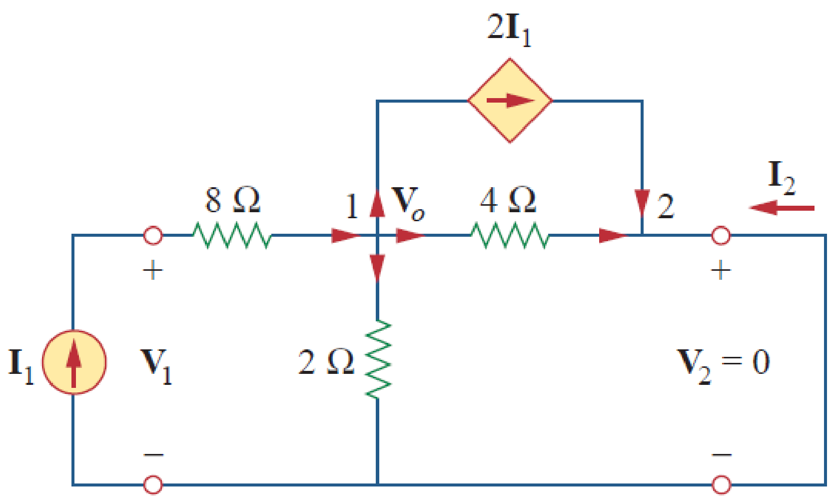

Figure 12.8 Short circuit output condition

The KCL node equation for node 1 is in (12.28).

$$\frac{\boldsymbol{V}_1 - \boldsymbol{V}_0}{8} = 2\boldsymbol{I}_1 + \frac{\boldsymbol{V}_0}{2} + \frac{\boldsymbol{V}_0 - 0}{4} \tag{12.28}$$

Because I_1 flows through the 8-Ω resistor we can substitute this condition into (12.28).

$$\boldsymbol{I}_1 = \frac{\boldsymbol{V}_1 - \boldsymbol{V}_0}{8} \rightarrow \frac{\boldsymbol{V}_1 - \boldsymbol{V}_0}{8} = 2\frac{\boldsymbol{V}_1 - \boldsymbol{V}_0}{8} + \frac{3\boldsymbol{V}_0}{4} \tag{12.29}$$

After sorting and simplifying (12.29) we can write the following.

$$0 = \frac{\boldsymbol{V}_1 - \boldsymbol{V_0}}{8} + \frac{3\boldsymbol{V}_0}{4} \tag{12.30}$$

$$0 = \boldsymbol{V}_1 - \boldsymbol{V_0} + 6\boldsymbol{V}_0 \rightarrow \boldsymbol{V}_1 = -5\boldsymbol{V}_0 \tag{12.31}$$

Substituting V_1 from (12.31) into the (12.29) input condition

$$\boldsymbol{I}_1 = \frac{\boldsymbol{V}_1 - \boldsymbol{V_0}}{8} = \frac{-5\boldsymbol{V_0} - \boldsymbol{V_0}}{8} = -0.75\boldsymbol{V_0} \tag{12.32}$$

And finally, the short circuit input admittance parameter is

$$\boldsymbol{y}_{11} = \frac{\boldsymbol{I}_1}{\boldsymbol{V}_1} = \frac{-0.75\boldsymbol{V_0}}{-5\boldsymbol{V}_0} = 0.15 \text{ S} \tag{12.33}$$

Now, we continue with a calculation of the y_{21} transfer admittance. From (12.26) the characteristic equation y_{21} is expressed with a short circuit condition as in (12.34).

$$\boldsymbol{y}_{21} = \left.\frac{\boldsymbol{I}_2}{\boldsymbol{V}_1}\right|_{\boldsymbol{V}_2=0} \tag{12.34}$$

The circuit we examine is the same as it was in the y_{11} calculation because the short circuit output is the same condition. Thus (12.31) and (12.32) are still valid calculations for V_1 and I_1.

Applying the KCL for node 2 in the circuit shown in Fig. 12.8 gives us (12.35).

$$\frac{\boldsymbol{V}_0 - 0}{4} + 2\boldsymbol{I}_1 + \boldsymbol{I}_2 = 0 \tag{12.35}$$

Substituting I_1 from (12.32) into (12.35)

$$\frac{\boldsymbol{V}_0}{4} + 2 \cdot (-0.75\boldsymbol{V}_0) = -\boldsymbol{I}_2 = -1.25\boldsymbol{V}_0 \tag{12.36}$$

thus,

$$\boldsymbol{y}_{21} = \frac{\boldsymbol{I}_2}{\boldsymbol{V}_1} = \frac{1.25\boldsymbol{V}_0}{-5\boldsymbol{V}_0} = -0.25\ S \tag{12.37}$$

The next admittance parameter we determine is the y_{12} transfer admittance.

$$\boldsymbol{y}_{12} = \left.\frac{\boldsymbol{I}_1}{\boldsymbol{V}_2}\right|_{\boldsymbol{V}_1=0} \tag{12.38}$$

The short circuit input condition is applied which gives us the circuit in Fig. 12.9.

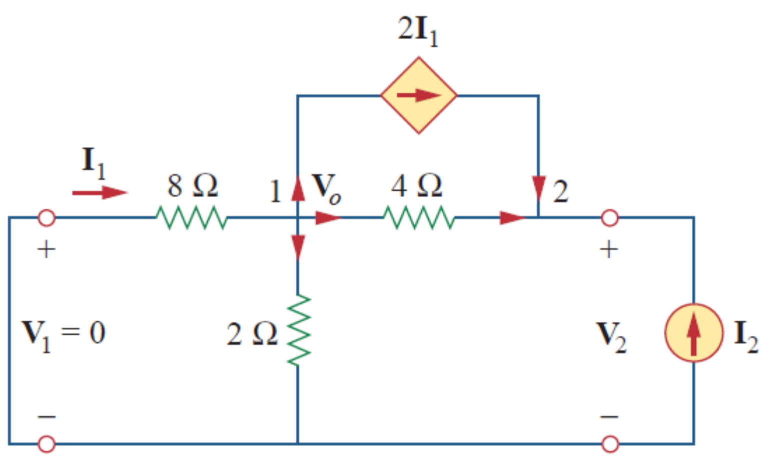

Figure 12.9 Short circuit input condition

Now, KCL for node 1 looks like this

$$\frac{0-\boldsymbol{V}_0}{8}=2\boldsymbol{I}_1+\frac{\boldsymbol{V}_0}{2}+\frac{\boldsymbol{V}_0-\boldsymbol{V}_2}{4} \tag{12.39}$$

And the input condition is the following

$$\boldsymbol{I}_1=\frac{0-\boldsymbol{V}_0}{8} \tag{12.40}$$

By substituting (12.40) into (12.39) we get

$$\frac{-\boldsymbol{V}_0}{8}=\frac{-2\boldsymbol{V}_0}{8}+\frac{\boldsymbol{V}_0}{2}+\frac{\boldsymbol{V}_0-\boldsymbol{V}_2}{4} \tag{12.41}$$

From which we can express voltage V_2 as in (12.42).

$$0=-\boldsymbol{V}_0+4\boldsymbol{V}_0+2\boldsymbol{V}_0-2\boldsymbol{V}_2 \rightarrow \boldsymbol{V}_2=2.5\boldsymbol{V}_0 \tag{12.42}$$

Finally, from (12.40) and (12.42) we have the transfer admittance we are looking for.

$$\boldsymbol{y}_{12}=\frac{\boldsymbol{I}_1}{\boldsymbol{V}_2}=\frac{-\boldsymbol{V}_0/8}{2.5\boldsymbol{V}_0}=-0.05\,S \tag{12.43}$$

The last admittance parameter, we have to find, is the short circuit output admittance.

$$\boldsymbol{y}_{22}=\left.\frac{\boldsymbol{I}_2}{\boldsymbol{V}_2}\right|_{\boldsymbol{V}_1=0} \tag{12.44}$$

We use the same circuit, shown in Fig. 12.9, because of the same short input condition, so (12.40) and (12.42) are still valid.

Applying KCL for node 2

$$\frac{\boldsymbol{V_0}-\boldsymbol{V}_2}{4}+2\boldsymbol{I_1}+\boldsymbol{I}_2=0 \tag{12.45}$$

And substituting V2 and I1 from (12.40) and (12.42) into (12.45)

$$-\boldsymbol{I}_2=\frac{\boldsymbol{V}_0-(2.5\boldsymbol{V}_0)}{4}-2\frac{\boldsymbol{V}_0}{8}=-0.625\boldsymbol{V}_0 \tag{12.46}$$

Thus,

$$\boldsymbol{y}_{22}=\frac{\boldsymbol{I}_2}{\boldsymbol{V}_2}=\frac{0.625\boldsymbol{V}_0}{2.5\boldsymbol{V}_0}=0.25\text{ S} \tag{12.47}$$

The admittance matrix for the circuit in Fig. 12.7 is the following.

$$\boldsymbol{Y}=\begin{bmatrix}0.15 & -0.05\\ -0.25 & 0.25\end{bmatrix}S \tag{12.48}$$

We see that $\boldsymbol{y}_{12}\neq\boldsymbol{y}_{21}$ so, the circuit is non-reciprocal, according to the dependent generator applied to it.

12.5 Transmission parameters

The transmission parameter set of the two-port network shown in Fig. 12.10 is given as the following

$$\boldsymbol{T} = \begin{bmatrix} 4 & 20\,\Omega \\ 0.1\,S & 2 \end{bmatrix} \tag{12.49}$$

The output port is connected to a variable load for maximum power transfer. Find R_L and the maximum power transferred.

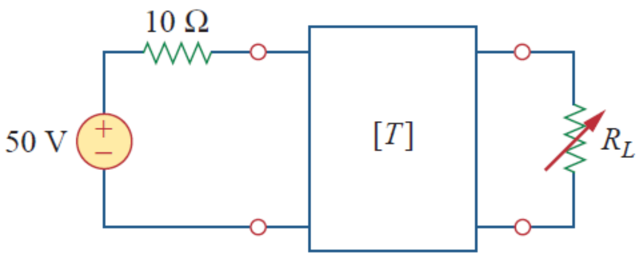

Figure 12.10 Two-port with transmission parameters

Solution

Our task is to determine the load resistance for maximum power transfer, so we have to determine the Thevenin equivalent circuit, connected to the load. So, the $\boldsymbol{Z}_{Th}$ impedance and $\boldsymbol{V}_{Th}$ voltage have to be calculated first.

Transmission characteristic equations are the following, with substitution of (12.49) the transmission matrix.

$$\boldsymbol{V}_1 = 4\boldsymbol{V}_2 - 20\boldsymbol{I}_2 \tag{12.50}$$

$$\boldsymbol{I}_1 = 0.1\boldsymbol{V}_2 - 2\boldsymbol{I}_2 \tag{12.51}$$

For Thevenin impedance calculation we first make the circuit energy-free by substituting the 50-V source with a short circuit. By applying this condition on the input side we can write (12.50) the following.

$$\boldsymbol{V}_1 = -10\boldsymbol{I}_1 \rightarrow -10\boldsymbol{I}_1 = 4\boldsymbol{V}_2 - 20\boldsymbol{I}_2 \tag{12.52}$$

From which I_1 current is

$$\boldsymbol{I}_1 = -0.4\boldsymbol{V}_2 + 2\boldsymbol{I}_2 \tag{12.53}$$

The right side of (12.50) is equal to the right side of (12.53). Thus,

$$0.1\boldsymbol{V}_2 - 2\boldsymbol{I}_2 = -0.4\boldsymbol{V}_2 + 2\boldsymbol{I}_2 \rightarrow 0.5\boldsymbol{V}_2 = 4\boldsymbol{I}_2 \tag{12.54}$$

From (12.54) we can express the Thevenin impedance.

$$\boldsymbol{Z}_{Th} = \frac{\boldsymbol{V}_2}{\boldsymbol{I}_2} = \frac{4}{0.5} = 8\,\Omega \tag{12.55}$$

Now we can determine the Thevenin voltage as the voltage measured between the open output terminals of the circuit, as shown in Fig. 12.11.

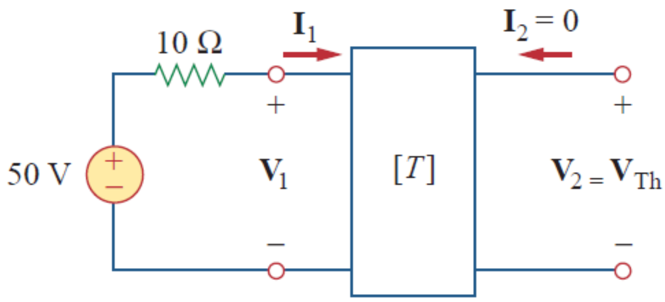

Figure 12.11 Circuit for Thevenin voltage calculation

According to the circuit in Fig. 12.11 the input and output conditions are given in (12.56). as the following.

$$I_2 = 0\,, V_1 = 50 - 10I_1 \tag{12.56}$$

The transmission parameters of the two-port circuit is given by (12.50) and (12.51) as the following. From (12.50) and (12.56) we can write

$$50 - 10I_1 = 4V_2 \tag{12.57}$$

And from (12.51) and (12.57) we can write

$$50 - V_2 = 4V_2 \rightarrow V_2 = V_{Th} = 10\ V \tag{12.58}$$

Thus, we have the Thevenin equivalent circuit as shown in Fig. 12.12.

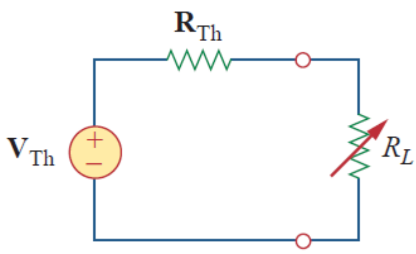

Figure 12.12 Thevenin equivalent circuit

For maximum power transfer the optimal load is given by (12.59).

$$R_L = Z_{Th} = R_{Th} = 8\ \Omega \tag{12.59}$$

The value of that maximum power is

$$P = I^2 \cdot R_L = \left(\frac{V_{Th}}{2R_L}\right)^2 \cdot R_L == \frac{V_{Th}^2}{4R_L} = \frac{100}{4 \cdot 8} = 3.125\ W \tag{12.60}$$

12.6 Two-port interconnections

Find the *y* parameters of the two-ports in Fig. 12.13.

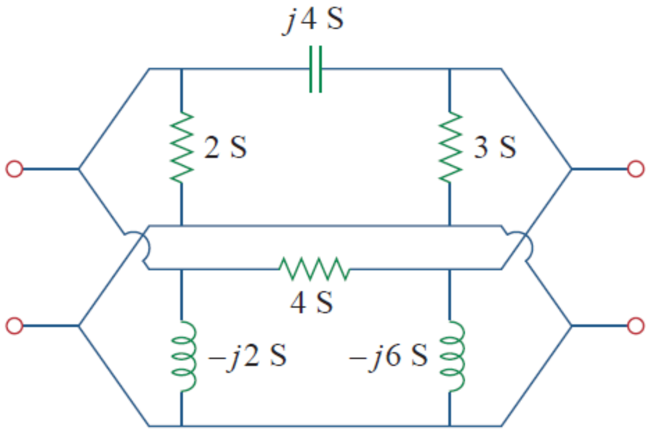

Figure 12.13 Parallel connected two-ports

Solution

The admittance characteristic is given by (12.61) and (12.62).

$$\boldsymbol{I}_1 = \boldsymbol{y}_{11}\boldsymbol{V}_1 + \boldsymbol{y}_{12}\boldsymbol{V}_2 \tag{12.61}$$

$$\boldsymbol{I}_2 = \boldsymbol{y}_{21}\boldsymbol{V}_1 + \boldsymbol{y}_{22}\boldsymbol{V}_2 \tag{12.62}$$

As shown in Fig. 12.13, the ∏-section circuits are connected in parallel. The admittance matrix of each is given in (12.63) and (12.64).

$$\boldsymbol{Y}_a = \begin{bmatrix} 2 + j4 & -j4 \\ -j4 & 3 + j4 \end{bmatrix} \tag{12.63}$$

$$\boldsymbol{Y}_b = \begin{bmatrix} 4 - j2 & -4 \\ -4 & 4 - j6 \end{bmatrix} \tag{12.64}$$

Because of the parallel connection, the equivalent admittance matrix is the addition of the component admittance matrixes.

$$\boldsymbol{Y} = \boldsymbol{Y}_a + \boldsymbol{Y}_b \tag{12.65}$$

Thus,

$$\boldsymbol{Y} = \begin{bmatrix} 2 + j4 & -j4 \\ -j4 & 3 + j4 \end{bmatrix} + \begin{bmatrix} 4 - j2 & -4 \\ -4 & 4 - j6 \end{bmatrix} = \begin{bmatrix} 6 + j2 & -4 - j4 \\ -4 - j4 & 7 - j2 \end{bmatrix} \tag{12.66}$$

12.7 Wave impedance

Calculate the wave impedance of the symmetric T-section shown in Fig. 12.14.

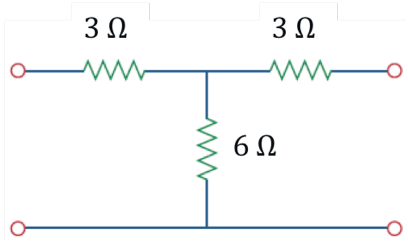

Figure 12.14 T-section circuit

Solution 1

According to the definition of wave impedance we have to determine the short circuit and open circuit input impedance.

$$\boldsymbol{Z}_{SC} = 3 + 3 \times 6 = 3 + \frac{3 \cdot 6}{3+6} = 5\,\Omega \tag{12.67}$$

$$\boldsymbol{Z}_{OC} = 3 + 6 = 9\,\Omega \tag{12.68}$$

Thus, wave impedance is

$$\boldsymbol{Z}_0 = \sqrt{\boldsymbol{Z}_{SC} \cdot \boldsymbol{Z}_{OC}} = \sqrt{45} = 3\sqrt{5}\,\Omega \tag{12.69}$$

Solution 2

If the symmetric two-port is terminated by its wave impedance, the input impedance will also be equal to the wave impedance. Thus, another way for calculation is in (12.70).

$$\boldsymbol{Z}_{IN} = 3 + 6 \times (3 + \boldsymbol{Z}_0) = \boldsymbol{Z}_0 \to 3 + \frac{18 + 6\boldsymbol{Z}_0}{9 + \boldsymbol{Z}_0} = \boldsymbol{Z}_0 \tag{12.70}$$

When eliminating the denominator in (12.70) we get the following.

$$27 + 3\boldsymbol{Z}_0 + 18 + 6\boldsymbol{Z}_0 = 9\boldsymbol{Z}_0 + \boldsymbol{Z}_0^2 \tag{12.71}$$

$$45 = \boldsymbol{Z}_0^2 \to \boldsymbol{Z}_0 = \sqrt{45} = 3\sqrt{5}\,\Omega \tag{12.72}$$

This calculation gives the same result as we already determined in (12.69)

12.8 Bartlett's bisection theorem

Calculate the wave impedance and Z parameters of the symmetric T-section using Bartlett's theorem. The T-section circuit is given in Fig. 12.14.

Solution

When applying Bartlett's bisection theorem, we can simplify things by using the half-network as shown in Fig. 12.15.

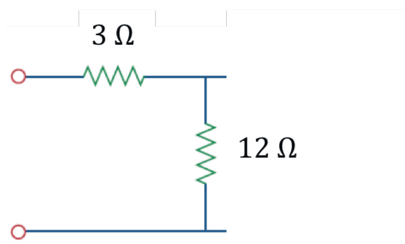

Figure 12.15 Half-network of the T-section

The short circuit input impedance of the half-network is

$$\boldsymbol{Z}_I = \boldsymbol{Z}_{SC-half} = (\boldsymbol{Z}_{11} - \boldsymbol{Z}_{12}) = 3\,\Omega \tag{12.73}$$

The open circuit input impedance of the half-network is

$$\boldsymbol{Z}_{II} = \boldsymbol{Z}_{OC-half} = (\boldsymbol{Z}_{11} + \boldsymbol{Z}_{12}) = 3 + 12 = 15\Omega \tag{12.74}$$

The wave impedance of the original circuit is

$$\boldsymbol{Z}_0 = \sqrt{\boldsymbol{Z}_I \cdot \boldsymbol{Z}_{II}} = \sqrt{3 \cdot 15} = 3\sqrt{5}\,\Omega \tag{12.75}$$

(Please refer to the result in previous example, where the wave impedance was calculated another way, but with the same result.)

With the addition and subtraction of (12.73) and (12.74) we can express the input and transfer impedances.

$$\boldsymbol{Z}_{11} = \boldsymbol{Z}_{22} = \frac{\boldsymbol{Z}_{II} + \boldsymbol{Z}_I}{2} = \frac{15 + 3}{2} = 9\,\Omega \tag{12.76}$$

$$\boldsymbol{Z}_{12} = \boldsymbol{Z}_{21} = \frac{\boldsymbol{Z}_{II} - \boldsymbol{Z}_I}{2} = \frac{15 - 3}{2} = 6\,\Omega \tag{12.77}$$

The impedance matrix of the symmetric (and reciprocal) T-section circuit is given in (12.78).

$$\boldsymbol{Z} = \begin{bmatrix} 9 & 6 \\ 6 & 9 \end{bmatrix}\,\Omega \tag{12.78}$$

13. First-Order Dynamic Circuits

13.1 Charged capacitor

Calculate the insulation resistance of a 220 nF capacitor if the voltage of the charged capacitor decays into its third part in 10 mins.

Solution

The voltage of a charged capacitor decays according to its loss resistance as described in (13.1).

$$v(t) = V\,e^{-\frac{t}{\tau}}, \qquad \tau = RC \tag{13.1}$$

Voltage decays into its third part in 10 mins, i.e. in 600 s thus,

$$\frac{V}{3} = V\,e^{-\frac{600}{\tau}} \rightarrow \tau = RC = \frac{600}{\ln 3} \tag{13.2}$$

Expressing the loss resistance from (13.2) we get the following

$$R = \frac{600}{C \cdot \ln 3} = \frac{600}{220 \cdot 10^{-9} \cdot \ln 3} = 2.48 \cdot 10^{9} = 2.48\ G\Omega \tag{13.3}$$

13.2 Source-free RC circuit

Find $v_C(t),\ v_X(t),\ i_X(t)$ for $t \geq 0$ if in the circuit shown in Fig. 13.1. Assume that the initial voltage across the capacitor is $v_C(0) = 15\ V$

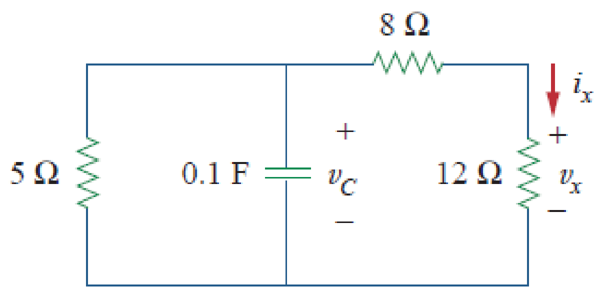

Figure 13.1 Source-free RC circuit

Solution

For the time constant we need to know the equivalent resistance, connected to the capacitor as seen in Fig. 13.2.

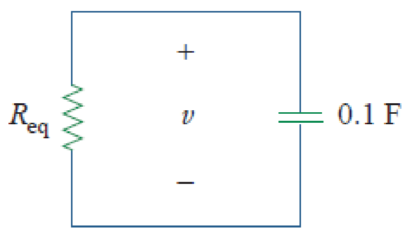

Figure 13.2 Equivalent circuit for Fig. 13.1

The equivalent resistance is given in (13.4)

$$R_{eq} = 20 \times 5 = 4\,\Omega \tag{13.4}$$

thus, the time constant is

$$\tau = R_{eq}C = 4 \cdot 0.1 = 0.4\ s \tag{13.5}$$

Voltage across the capacitor is given by (13.6).

$$v(t) = v(0)\, e^{-\frac{t}{\tau}} = 15\, e^{-\frac{t}{0.4}} = 15\, e^{-2.5t}\ V \tag{13.6}$$

By applying voltage division for the circuit in Fig. 13.1, we can calculate voltage v_X.

$$v_X = \frac{12}{12+8} v = 0.6 \cdot 15\, e^{-2.5t} = 9\, e^{-2.5t}\ V \tag{13.7}$$

from which, the current i_X, flowing through the 12-Ω resistor is

$$i_X = \frac{v_X}{12} = 0.75\, e^{-2.5t}\ A \tag{13.8}$$

13.3 RC circuit with switch

In the circuit, shown in Fig. 13.3, the switch has been closed for a long time, and is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

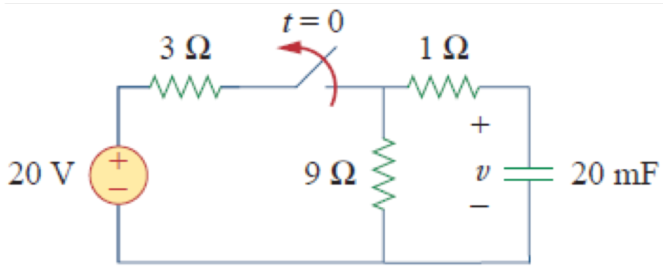

Figure 13.3 RC circuit with switch

Solution

We have to first determine the initial condition but because the switch has been closed for a long time, the voltage across the capacitor is obtained from (13.9). No initial current is measured through the capacitor according to the stationery (DC) condition. Please see the Fig. 13.4 for calculating the initial condition.

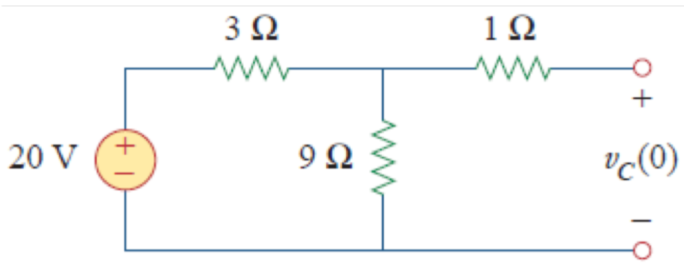

Figure 13.4 Circuit for finding initial condition

$$v_C(-0) = 20 \cdot \frac{9}{9+3} = 15\ V = v_C(+0) = V_0 \tag{13.9}$$

With the switch is open at t = 0, the circuit is shown in Fig. 13.5 with the initial voltage of 15 V across the capacitor. The discharging transient procedure will go through the resistance, as calculated in (13.10).

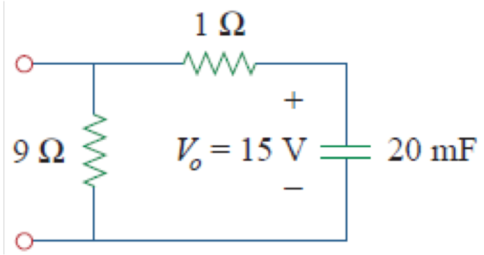

Figure 13.5 Circuit when the switch is open

$$R_{eq} = 1 + 9 = 10\,\Omega \tag{13.10}$$

The time constant for the transient is

$$\tau = R_{eq}C = 10 \cdot 20 \cdot 10^{-3} = 0.2\,s \tag{13.11}$$

and the voltage across the capacitor for t ≥ 0 is

$$v(t) = v_C(0)\,e^{-\frac{t}{\tau}} = 15e^{-\frac{t}{0.2}} = 15e^{-5t}\,V \tag{13.12}$$

Finally, the energy, stored in the capacitor is calculated in (13.13).

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \cdot 20 \cdot 10^{-3} \cdot 15^2 = 2.25\,J \tag{13.13}$$

13.4 Source-free RL circuit

Assuming that *i(0)* = 10 A calculate *i(t)* and $i_x(t)$ for the circuit in Fig. 13.6.

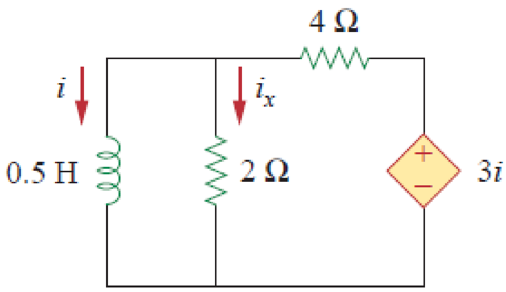

Figure 13.6 Source-free RL circuit

Solution

Method 1 - by finding Thevenin equivalent resistance.

Because the circuit contains a dependent source, the equivalent resistance between terminal (a) and terminal (b) is determined according to the Fig. 13.7. Supposing i_1 and i_2 mesh currents the mesh equations are shown in (13.14) and (13.15).

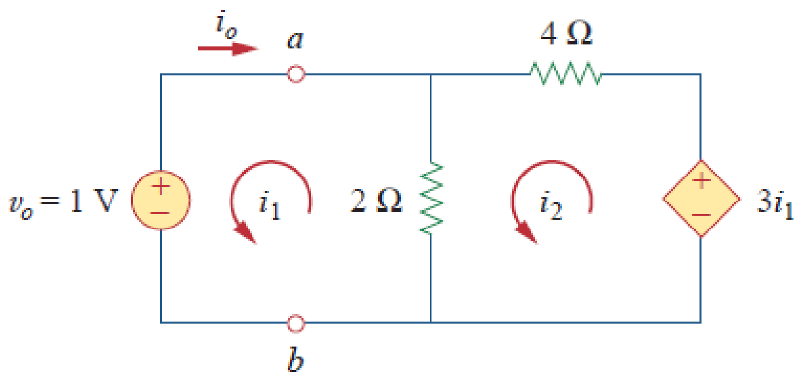

Figure 13.7 Circuit for finding R_{Th} resistance

$$2(i_1 - i_2) + 1 = 0 \rightarrow i_1 - i_2 = -\frac{1}{2} \tag{13.14}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \rightarrow i_2 = \frac{5}{6} i_1 \tag{13.15}$$

Solving the equations, we find the i_0 current as the following.

$$i_1 = -3\ A\,, \qquad i_0 = -i_1 = 3\ A \tag{13.16}$$

Thus, the equivalent (Thevenin) resistance and the time constant of the circuit can be calculated as in (13.17).

$$R_{eq} = R_{Th} = \frac{v_0}{i_0} = \frac{1}{3}\ \Omega \rightarrow \tau = \frac{L}{R_{eq}} = \frac{3}{2}\ s \tag{13.17}$$

The current, through the inductor is

$$i(t) = i(0)\, e^{-\frac{t}{\tau}} = 10\, e^{-\frac{2}{3}t}\ A\,, \qquad t \geq 0 \tag{13.18}$$

Voltage across the 2-Ω resistor is the same as the voltage through the inductor. Current i_x is given by Ohm's law applying the calculation of the voltage across the inductor.

$$i_x(t) = \frac{v_L(t)}{R} = \frac{L\, {}^{di}/_{dt}}{R} = -\frac{5}{3}\, e^{-\frac{2}{3}t}\ A\,, \qquad t \geq 0 \tag{13.19}$$

Method 2 – by applying Kirchhoff Voltage Law.

For this method we use the mesh currents as given in Fig. 13.8.

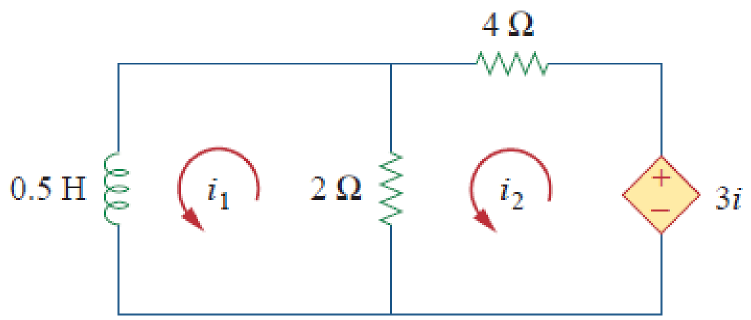

Figure 13.8 The RL circuit with mesh currents

The mesh equations are the following.

$$0.5\frac{di_1}{dt} + 2(i_1 - i_2) = 0 \rightarrow \frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \tag{13.20}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \rightarrow i_2 = \frac{5}{6}i_1 \tag{13.21}$$

Substituting i_2 from (13.21) into (13.20) we have the following homogeneous differential equation.

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0 \tag{13.22}$$

By rearranging (13.22) we can write it as in (13.23).

$$\frac{di_1}{i_1} = -\frac{2}{3}dt = 0 \rightarrow \ln i|_{i(0)}^{i(t)} = -\frac{2}{3}t\Big|_0^t \tag{13.23}$$

thus,

$$\ln\frac{i(t)}{i(0)} = -\frac{2}{3}t \tag{13.24}$$

and expressing i(t) from (13.24) we have the time varying current through the inductor.

$$i(t) = i(0)\, e^{-\frac{t}{\tau}} = 10\, e^{-\frac{2}{3}t}\, A\,, \qquad t \geq 0 \tag{13.25}$$

The current through the 2-Ω resistor is

$$i_x(t) = \frac{v_L(t)}{R} = \frac{L\, {di}/{dt}}{R} = -\frac{5}{3}\, e^{-\frac{2}{3}t}\, A\,, \qquad t \geq 0 \tag{13.26}$$

The results given by method 1 and method 2 are the same.

13.5 RL Circuit with Switch

Find the *i(t)* current in the circuit shown in Fig. 13.9 for *t* > 0 time if the switch is open at t = 0 after a long closed time.

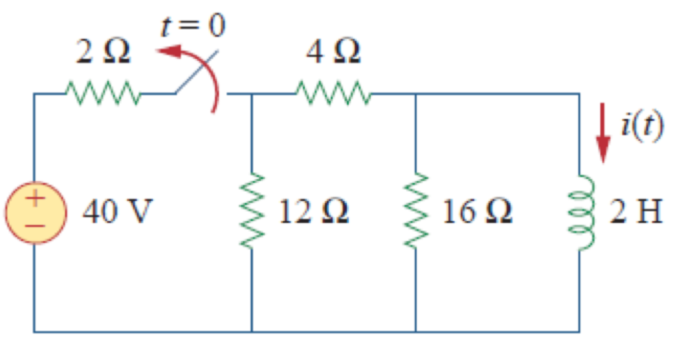

Figure 13.9 RL circuit with switch

Solution

The switch is closed for t < 0 times and the inductor is a short circuit in its stationary state. The circuit is shown in Fig. 13.10.

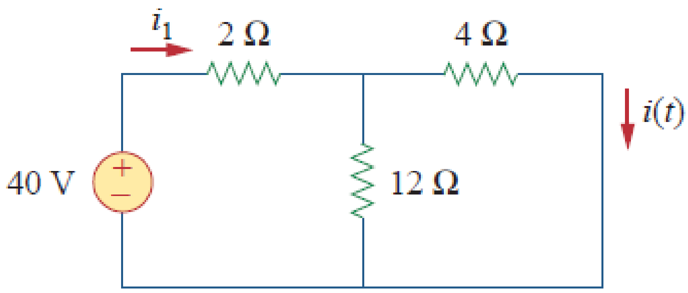

Figure 13.10 Circuit with the closed switch

For i_1 we need the equivalent resistance of the circuit, connected to the 40-V source.

$$4 \times 12 = \frac{4 \cdot 12}{4 + 12} = 3\,\Omega \rightarrow i_1 = \frac{40}{2 + 3} = 8\,A \tag{13.27}$$

Applying the current division, we can find the inductors current for t < 0.

$$i(0^-) = \frac{12}{12 + 4} i_1 = 6\,A\ (t < 0) \tag{13.28}$$

According to the property of the inductor, this current has no singularity.

$$i(0) = i(0^-) = 6\,A \tag{13.29}$$

Thus, the circuit is shown in Fig. 13.11 when the switch is open.

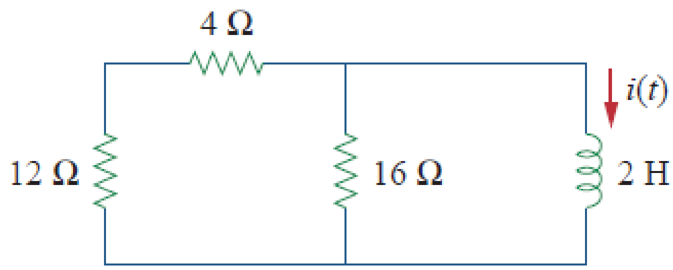

Figure 13.11 Circuit with open switch

The equivalent resistance and the time constant of the circuit for t > 0 is calculated in (13.30).

$$R_{eq} = (12 + 4) \times 16 = 8\,\Omega \rightarrow \tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4}\,s \tag{13.30}$$

And finally, the time function of the current flowing through the circuit is given in (13.31)

$$i(t) = i(0)\, e^{-\frac{2}{3}t} = 6\, e^{-4\,t}\, A\,,\ \ t > 0 \tag{13.31}$$

13.6 Singularity functions

Express the voltage pulse, shown in Fig. 13.12, in terms of the unit step. Calculate its derivative and sketch it.

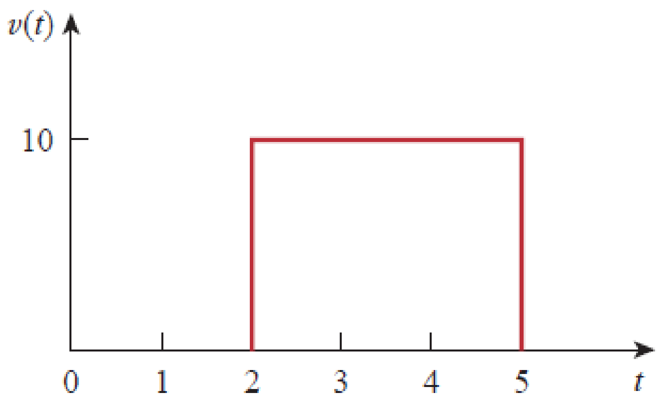

Figure 13.12 Voltage pulse

Solution

The voltage pulse, shown in Fig. 13.12 can be expressed as the superposition (subtraction) of the pair of unit step functions. Both of them are shifted in time i.e. they have their singularities at t = 2 s and t = 5 s. The *v(t)* function is given in (13.32).

$$v(t) = 10\,u(t-2) - 10\,u(t-5) = 10\,[u(t-2) - (t-5)] \qquad (13.32)$$

The derivative of the unit step is the unit impulse (or Dirac impulse). Thus, the derivative of the given voltage pulse is given in (13.33).

$$\frac{dv}{dt} = 10\,[\delta(t-2) - \delta(t-5)] \qquad (13.33)$$

This function is shown in Fig. 13.13.

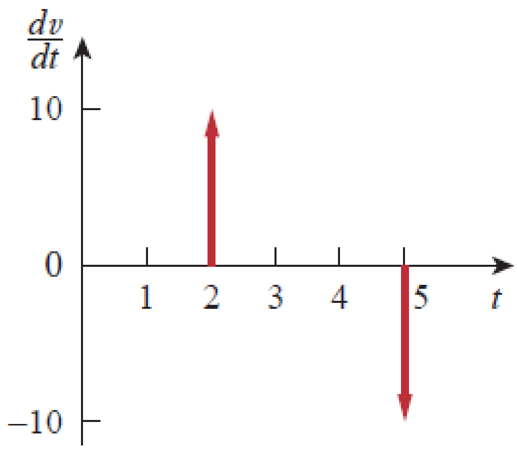

Figure 13.13 The derivative of the voltage pulse function

13.7 Step response of an RC circuit

The switch has been in position ‘A’ for a long time in the circuit, shown in Fig. 13.14. At t = 0 the switch moves to position ‘B’. Determine *v(t)* voltage function for t > 0 and calculate its value at t = 1 s and t = 4 s.

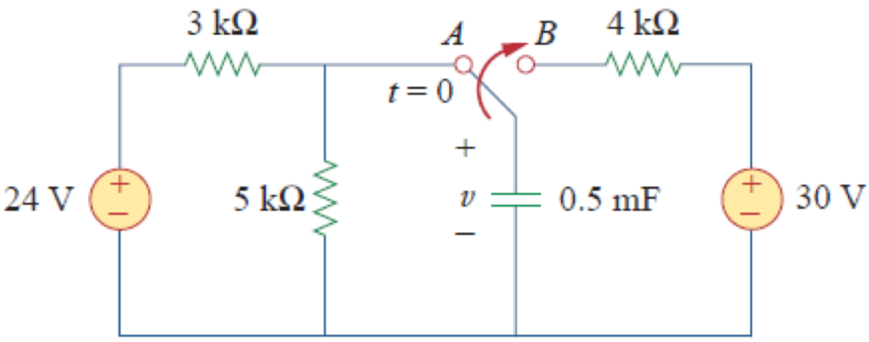

Figure 13.14 RC circuit for step response calculation

Solution

When the switch is in position A, the voltage across the capacitor can be calculated by applying the voltage division. According to the property of the capacitor this voltage has no singularity. The result is given in (13.34).

$$v(0^-) = 24 \cdot \frac{5}{5+3} = 15\,V = v(0^+) \tag{13.34}$$

When the switch is in position B for an extended period, the circuit is in its second stationary state. The voltage across the capacitor is given in (13.35).

$$v(\infty) = 30\,V \tag{13.35}$$

The time constant of the circuit for t > 0 is given in (13.36).

$$\tau = R_{Th}C = 4 \cdot 10^3 \cdot 0.5 \cdot 10^{-3} = 2\,s \tag{13.36}$$

The step response of a first-order circuit is described in (13.37).

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \tag{13.37}$$

Thus, the voltage across the capacitor is

$$v(t) = 30 + (15 - 30)e^{-\frac{t}{2}} = (30 - 15e^{-0.5t})\,V \tag{13.38}$$

The voltage at t = 1 s is

$$v(1) = 30 - 15\,e^{-0.5} = 20.9\,V \tag{13.39}$$

The voltage at t = 4 s is

$$v(4) = 30 - 15\,e^{-2} = 27.97\,V \tag{13.40}$$

13.8 Step response of an RL circuit

Find *i(t)* for t > 0 if the switch in the circuit, shown in Fig. 13.15, has been previously closed for a long time.

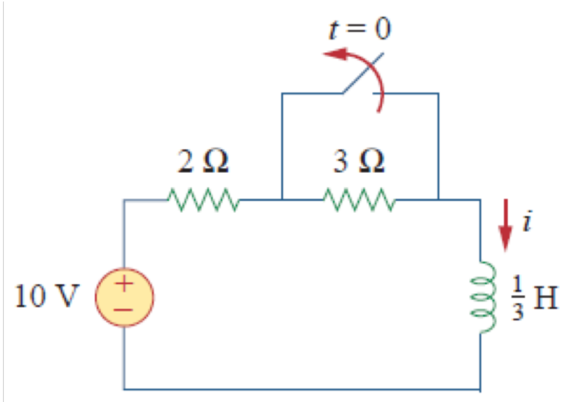

Figure 13.15 RL circuit for step response calculation

Solution

The switch is closed for t < 0 thus, the circuit current is given in (13.41).

$$i(0^-) = \frac{10}{2} = 5\,A = i(0^+) \tag{13.41}$$

The current in its second stationery state, i.e. when the switch is open for an extended period, is calculated as the following.

$$i(\infty) = \frac{10}{2+3} = 2\,A \tag{13.42}$$

The equivalent resistance of the circuit for t > 0 is

$$R_{Th} = 2 + 3 = 5\,\Omega \tag{13.43}$$

and the time constant is

$$\tau = \frac{L}{R_{Th}} = \frac{1/3}{5} = \frac{1}{15}\,s \tag{13.44}$$

The step response of a first-order circuit is given by the equation (13.45).

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \tag{13.45}$$

Substituting the circuit parameters and the initial conditions

$$i(t) = 2 + (5-2)e^{-15t} = 2 + 3e^{-15t}\,A,\ \ t > 0 \tag{13.46}$$

The result, obtained in (13.46) can be checked by applying Kirchhoff's Voltage Law for the single loop in the circuit for t > 0, as the following.

$$10 = 5i + L\frac{di}{dt} \tag{13.47}$$

Substituting the current from (13.46) into (13.47) we have (13.48).

$$10 = [10 + 15e^{-15t}] + \left[\frac{1}{3}\cdot 3\cdot(-15)e^{-15t}\right] \tag{13.48}$$

That is 10 = 10. (true).

14. Second-Order Dynamic Circuits

14.1 Initial and final conditions

Find the requested initial and steady-state values as the key to work conditions for the second-order dynamic circuit in Fig. 14.1. The switch was closed for an extended period and is open at t = 0.

$$(\boldsymbol{a}): i(0^+) =?, v(0^+) =?, (\boldsymbol{b}): \frac{di(0^+)}{dt} =?, \frac{dv(0^+)}{dt} =?, (\boldsymbol{c}): i(\infty) =?, v(\infty) =?$$

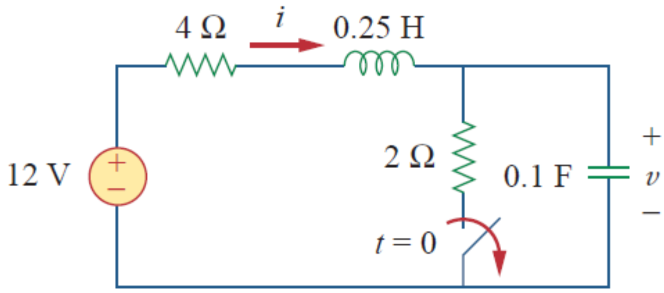

Figure 14.1 Second-order dynamic circuit

Solution (a)

The circuit was in its original steady state when the switch was in a closed position as seen in Fig. 14.2.

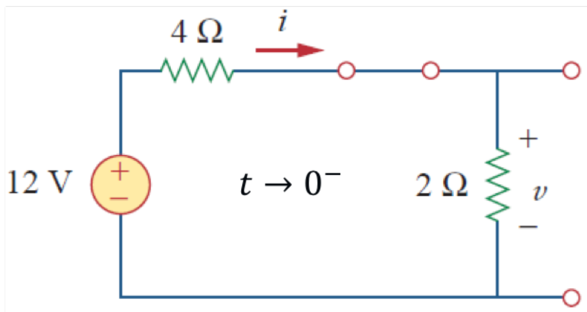

Figure 14.2 Steady state circuit when the switch is closed

The circuit current and *v* voltage at the closed switch is given by Ohm's law.

$$i(0^-) = \frac{12}{4+2} = 2\,A, \qquad v(0^-) = 2i = 4\,V \tag{14.1}$$

Because this current is also the inductors current, then

$$i(0^+) = i(0^-) = 2\,A, \qquad v(0^+) = v(0^-) = 4\,V \tag{14.2}$$

Solution (b)

When the switch is open the inductor's current flows through the capacitor.

$$i(0^+) = i_C(0^+) = 2\,A \tag{14.3}$$

We can express the first derivative of the voltage from the capacitor's properties, as given in (14.4).

$$i_C = C\frac{dv}{dt} \rightarrow \frac{dv}{dt} = \frac{i_C}{C} \tag{14.4}$$

Thus,

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\frac{V}{s} \tag{14.5}$$

We have to also determine the first derivative of the current. It can be expressed from the inductors properties, as given in (14.6).

$$v_L = L\frac{di}{dt} \rightarrow \frac{di}{dt} = \frac{v_L}{L} \tag{14.6}$$

The equivalent circuit, when the switch is open, is given in Fig. 14.3. Applying Kirchhoff's Voltage Law, we get (14.7).

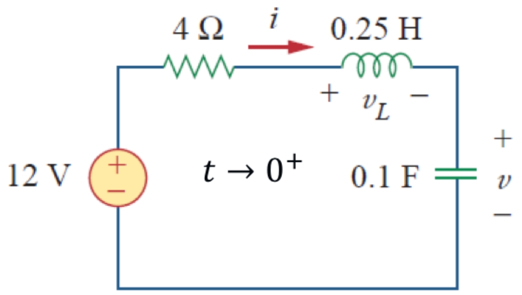

Figure 14.3 Equivalent circuit when the switch is open

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0 \tag{14.7}$$

Expressing the inductors voltage from (14.7) we get the following.

$$v_L(0^+) = 12 - 8 - 4 = 0 \tag{14.8}$$

Thus,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\frac{\text{A}}{\text{s}} \tag{14.9}$$

Solution (c)

The steady state values are given according to the equivalent circuit, as shown in Fig. 14.4.

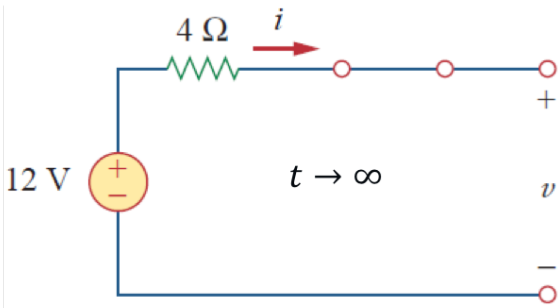

Figure 14.4 Steady state equivalent circuit

The electric current through the open circuit is zero and thus, the voltage across the open terminals is 12 V.

$$i(\infty) = 0, \qquad v(\infty) = 12V \tag{14.10}$$

14.2 The source-free RLC circuit

Find *i(t)* in the circuit, shown in Fig. 14.5, for $t > 0$. Steady-state has been reached at $t = 0^-$.

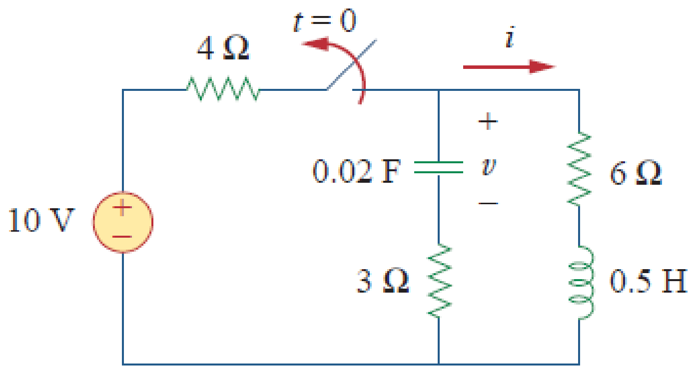

Figure 14.5 Second-order circuit

Solution

The steady state equivalent circuit while the switch is closed is given in Fig. 14.6.

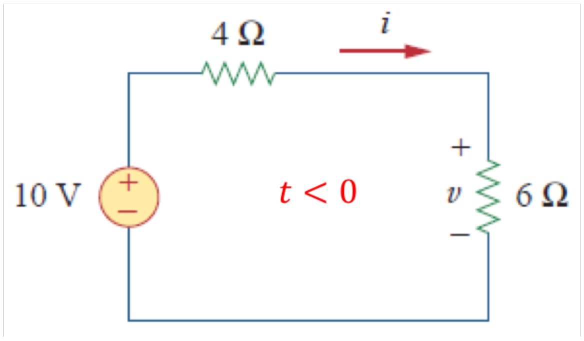

Figure 14.6 Equivalent circuit for $t < 0$

Thus, the initial current and voltage is given by Ohm's law.

$$i(0) = \frac{10}{4+6} = 1\,A, \qquad v(0) = 6i = 6\,V \tag{14.11}$$

The equivalent circuit shown in Fig. 14.7, when the switch is open.

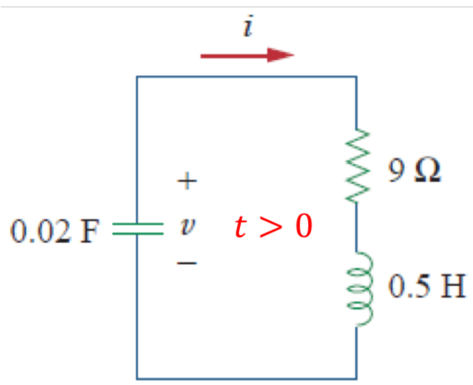

Figure 14.7 Equivalent circuit for $t > 0$

The damping factor and natural frequency are obtained from the circuit parameters as given in (14.12).

$$\delta = \frac{R}{2L} = \frac{9}{2 \cdot 0.5} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \cdot 0.02}} = 10 \tag{14.12}$$

First of all, we note that the circuit is underdamped according to (14.13).

$$\delta < \omega \rightarrow underdamped \tag{14.13}$$

The characteristic roots are the following.

$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100} \tag{14.14}$$

$$s_{1,2} = -9 \pm j4.359, \tag{14.15}$$

For an underdamped circuit the electric current is given in (14.16).

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \tag{14.16}$$

The A_1 and A_2 constants are obtained from the initial conditions as the following.

$$i(0) = 1 = e^0(A_1 \cos 0 + A_2 \sin 0) = A_1 \tag{14.17}$$

Applying KVL for the circuit in Fig. 14.7 gives us the following equation.

$$\frac{di(0)}{dt} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9 \cdot 1 - 6] = -6\frac{A}{s} \tag{14.18}$$

The first derivative of (14.16) is given in (14.19).

$$\begin{aligned} \frac{di(t)}{dt} &= -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \ldots \\ &\ldots + e^{-9t}4.359(-A_1 \sin 4.359t + A_2 \cos 4.359t) \end{aligned} \tag{14.19}$$

For $t = 0$ we can write the following form (14.18) and (14.19).

$$-6 = -9(A_1 \cos 0 + A_2 \sin 0) + 4.359(-A_1 \sin 0 + A_2 \cos 0) \tag{14.20}$$

$$-6 = -9A_1 + 4.359A_2 \tag{14.21}$$

Thus,

$$A_1 = 1 \rightarrow -6 = -9 + 4.359A_2 \rightarrow A_2 = 0.6882 \tag{14.22}$$

Finally, the requested current is given in (14.23)

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t)\ A \tag{14.23}$$

14.3 Step response of the RLC circuit

Find $v(t)$ and $i(t)$ in Fig. 14.8 for $t > 0$. Consider the cases R=5 Ω, R=4 Ω, R=1 Ω.

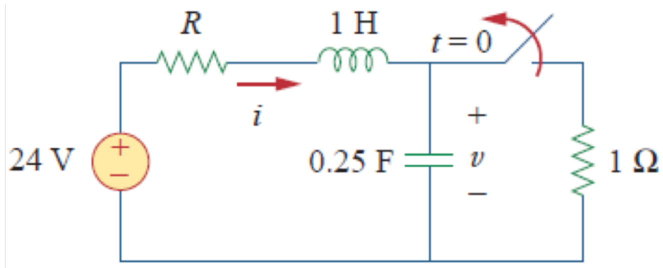

Figure 14.8 RLC circuit

Solution - Case R = 5 Ω

The initial conditions of *i(0)* and *v(0)* can be calculated according to Ohm's law.

$$i(0) = \frac{24}{5+1} = 4\,A,\ v(0) = 1 \cdot i = 4\,V \tag{14.24}$$

The damping factor and natural frequency are the following.

$$\delta = \frac{R}{2L} = \frac{5}{2 \cdot 1} = 2.5,\ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \cdot 0.25}} = 2 \tag{14.25}$$

Thus, the roots of the characteristic equation are given by (14.26). We have two real roots which means that the circuit has an overdamped natural response.

$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} = -1,\ -4 \tag{14.26}$$

The total response is obtained from (14.27) as the sum of the steady-state response and transient (natural) response.

$$v(t) = v_{SS} + (A_1 e^{-t} + A_2 e^{-4t}) \tag{14.27}$$

The steady-state response is seen in Fig. 14.8 as no DC current flows through the capacitor.

$$v_{SS} = 24\,V \tag{14.28}$$

A₁ and *A₂* parameters can be calculated from the initial conditions. From (14.24) and (14.27) and for t = 0 we can write the following.

$$v(0) = 4 = 24 + (A_1 e^0 + A_2 e^0) \rightarrow A_1 + A_2 = -20 \tag{14.29}$$

Substituting (14.24) initial value into the capacitors properties we get (14.30).

$$i(0) = C\frac{dv(0)}{dt} = 4 \rightarrow \frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16 \tag{14.30}$$

A derivation of (14.27) gives the following result.

$$\frac{dv}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t} \rightarrow \frac{dv(0)}{dt} = 16 = -A_1 - 4A_2 \tag{14.31}$$

Thus, solving (14.29) and (14.31) the A_1 and A_2 parameters gives the following.

$$A_1 = -\frac{64}{3}, \qquad A_2 = 4/3 \tag{14.32}$$

So, the results for *v(t)* and *i(t)* are given in (14.33) and (14.34).

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t})\ V \tag{14.33}$$

$$i(t) = C\frac{dv(t)}{dt} = \frac{4}{3}(4e^{-t} - e^{-4t})\ A \tag{14.34}$$

Note that $i(0) = 4\ A$ from (14.33) as it was expected from (14.24).

Solution - Case $\mathrm{R} = 4\ \Omega$

The initial conditions of *i(0)* and *v(0)* can be calculated according to Ohm's law.

$$i(0) = \frac{24}{4+1} = 4.8\ A\,, \qquad v(0) = 1 \cdot i = 4.8V \tag{14.35}$$

For $t > 0$ the circuit is a series RLC circuit, see Fig. 14.8. Thus, the damping factor and natural frequency are the following.

$$\delta = \frac{R}{2L} = \frac{4}{2 \cdot 1} = 2\,, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \cdot 0.25}} = 2 \tag{14.36}$$

Because the damping factor and natural frequency have the same values the roots of the characteristic equation are equal as given in (14.37). The circuit has a critically damped natural response.

$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} = -2 \tag{14.37}$$

The total response is given in (14.38). A steady-state response is independent of *R* resistance in this circuit as there is no steady-state current due to the series capacitance.

$$v(t) = v_{SS} + (A_1 + A_2 t)e^{-2t}\,, \qquad v_{SS} = 24\ V \tag{14.38}$$

To find A_1 and A_2 values we have to use the initial conditions in (14.38).

$$t = 0 \rightarrow v(0) = 4.8 = 24 + A_1 \rightarrow A_1 = -19.2 \tag{14.39}$$

Because *i* current flows through the capacitor we can write (14.40).

$$i(0) = C\frac{dv(0)}{dt} = 4.8 \rightarrow \frac{dv(0)}{dt} = \frac{4.8}{C} = \frac{4{,}8}{0.25} = 19.2 \tag{14.40}$$

Calculating the first derivative of (14.38), we get (14.41)

$$\frac{dv}{dt} = (-2A_1 - 2tA_2 + A_2)e^{-2t} \rightarrow \frac{dv(0)}{dt} = 19.2 = -2A_1 + A_2 \tag{14.41}$$

Thus, A_1 and A_2 are given from (14.39) and (14.41).

$$A_1 = A_2 = -19.2 \tag{14.42}$$

Thus, the requested voltage and current are the following

$$v(t) = 24 - 19.2(1 + t)e^{-2t}\ V \tag{14.43}$$

$$i(t) = C\frac{dv(t)}{dt} = (4.8 + 9.6t)e^{-2t}\ A \tag{14.44}$$

Note that $i(0) = 4.8\ A$ from (14.44) as is expected according to (14.35).

Solution – Case $\mathrm{R} = 1\ \Omega$

The initial conditions are the following in this case.

$$i(0) = \frac{24}{1+1} = 12\ A, v(0) = 1 \cdot i = 12\ V \tag{14.45}$$

The damping factor and natural frequency are given in (14.46).

$$\delta = \frac{R}{2L} = \frac{1}{2 \cdot 1} = 0.5\,, \qquad \omega_0 = 2 \tag{14.46}$$

Because $\delta < \omega_0$ the circuit has an undamped natural response with complex conjugate roots.

$$s_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} = -0.5 \pm j1.936 \tag{14.47}$$

Thus, the total response function of the requested voltage is given in (1.48).

$$v(t) = 24 + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t} \tag{14.48}$$

To find A_1 and A_2 values we have to use initial conditions in (14.48).

$$t = 0 \to v(0) = 12 = 24 + A_1 \to A_1 = -12 \tag{14.49}$$

Because *i* current flows through the capacitor we can write (14.50).

$$i(0) = C\frac{dv(0)}{dt} = 12 \to \frac{dv(0)}{dt} = \frac{12}{C} = \frac{12}{0.25} = 48 \tag{14.50}$$

Calculating the first derivative of (14.48), we get (14.51)

$$\begin{aligned}\frac{dv}{dt} = e^{-0.5t}(&-1.936A_1 \sin 1.936t + 1.936A_2 \cos 1.936t) \\ &- 0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t)\end{aligned} \tag{14.51}$$

Substituting values for *t = 0* we can write (14.52).

$$\to \frac{dv(0)}{dt} = 48 = (-0 + 1.936A_2) - 0.5(A_1 + 0) \tag{14.52}$$

Thus, A_1 and A_2 are given from (14.49) and (14.52).

$$A_1 = -12\,, \qquad A_2 = +21.694 \tag{14.53}$$

Finally, the requested voltage and current are the following

$$v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t}\ V \tag{14.54}$$

$$i(t) = C\frac{dv(t)}{dt} = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t}\ A \tag{14.55}$$

Note that $i(0) = 12\ A$ from (14.55) as is expected according to (14.45).

For illustration purposes the calculated *v(t)* voltage, in case of overdamped $(R = 5\ \Omega)$, critically damped $(R = 4\ \Omega)$, and underdamped $(R = 1\ \Omega)$, is shown in Fig. 14.9.

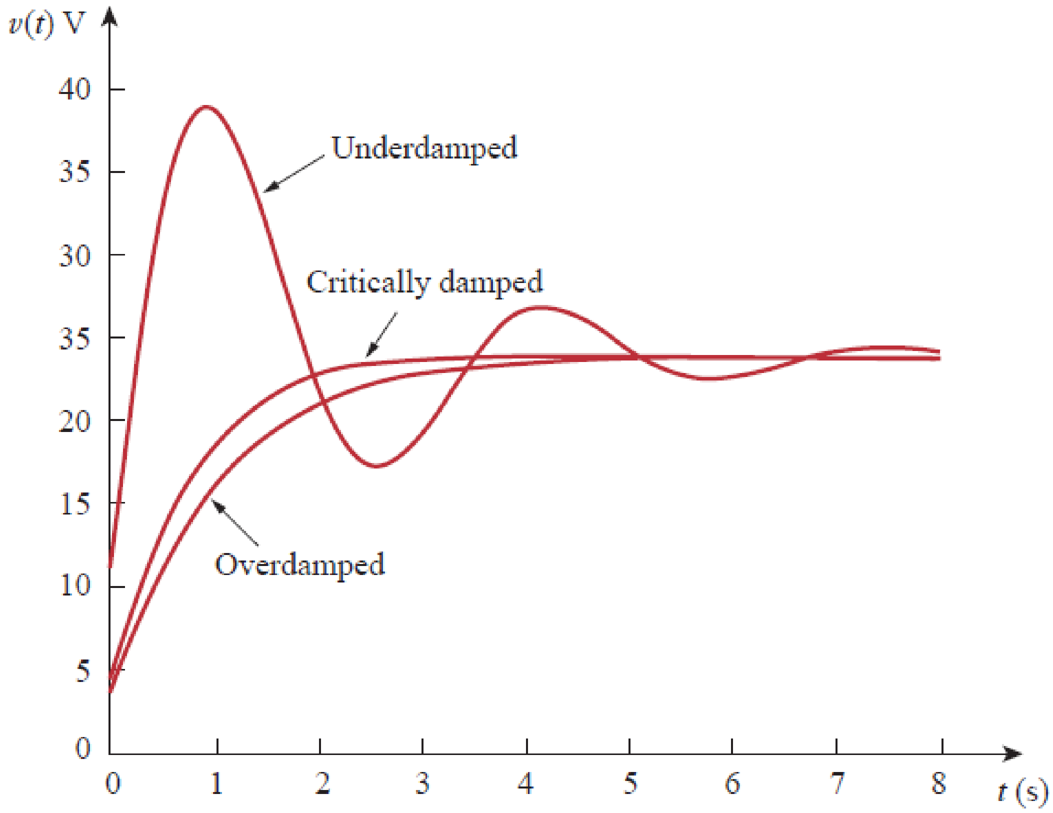

Figure 14.9 Overview of different damping cases

14.4 General second-order circuits

Find the complete response $v(t)$ and then $i(t)$ in the circuit shown in Fig. 14.10 for $t > 0$.

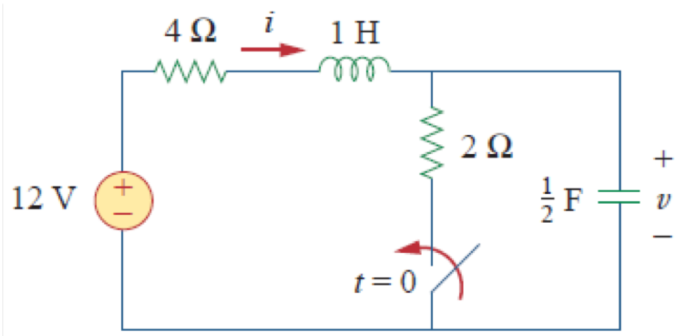

Figure 14.10 Example second order circuit

Solution

We can follow the general solution using the following five steps

Step 1 – Initial and final conditions

As voltage across the capacitor and current through the inductor must be continuous we can calculate these values as steady-state values for $t < 0$.

$$v(0^+) = v(0^-) = 12\,V, \qquad i(0^+) = i(0^-) = 0 \tag{14.56}$$

Initial value of the first derivative of the requested voltage is given in (14.57).

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} \tag{14.57}$$

When the switch is closed the circuit is given in Fig. 14.11.

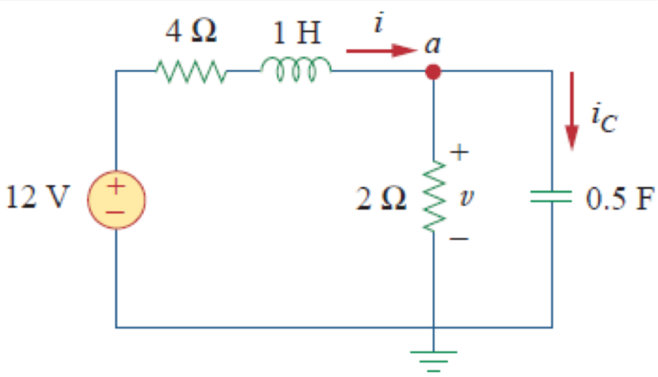

Figure 14.11 Circuit with closed switch

Applying KCL for node ‘a’ we can write (14.58).

$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2} \tag{14.58}$$

Substituting values from (14.56) into (14.58) the current through the capacitor gives

$$0 = i_C(0^+) + \frac{12}{2} \rightarrow i_C(0^+) = -6\ A \tag{14.59}$$

Thus,

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12\frac{V}{s} \tag{14.60}$$

By finding the steady-state values we can apply that inductor as a short circuit and the capacitor as an open circuit.

$$i(\infty) = \frac{12}{4+2} = 2\ A\,, \qquad v(\infty) = 2i(\infty) = 4\ V \tag{14.61}$$

Step 2 - Transient response (natural response for source-free circuit)

For the source-free circuit we apply the KCL at node ‘a’ and KVL for the mesh on the left side. See Fig. 14.11 when the independent source is eliminated i.e. replaced with a short circuit.

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \tag{14.62}$$

$$4i + 1\frac{di}{dt} + v = 0 \tag{14.63}$$

Substituting (14.62) into (14.63) we get (14.64)

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0 \tag{14.64}$$

that can also be written as

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0 \tag{14.65}$$

The characteristic equation of (14.65) is given in (14.66) with its roots.

$$s^2 + 5s + 6 = 0 \rightarrow s_{1,2} = -2\,, -3 \tag{14.66}$$

The circuit is overdamped in this case as it has two real roots thus, the transient response is given in (14.67).

$$v_{tr}(t) = A_1 e^{-2t} + A_2 e^{-3t} \tag{14.67}$$

Step 3 - Steady-state response

The steady-state value is calculated by applied voltage division according to the circuit in Fig. 14.11.

$$v_{ss}(t) = v(\infty) = 12\frac{2}{2+4} = 4\,V \tag{14.68}$$

Step 4 - Complete response

The complete response is given by the steady-state response and transient response as in (14.69).

(A_1 and A_2 are the coefficients to be determined in the next step.)

$$v(t) = v_{ss}(t) + v_{tr}(t) = 4 + A_1 e^{-2t} + A_2 e^{-3t} \tag{14.69}$$

Step 5 - Coefficients by applied initial values

When applying the initial value, as calculated in (14.56), and the steady-state value from (14.68), (14.69) for t = 0 we get the following.

$$v(0) = 12 = 4 + A_1 e^0 + A_2 e^0 \rightarrow 12 = 4 + A_1 + A_2 \rightarrow A_1 + A_2 = 8 \tag{14.70}$$

Substituting (14.60) into the first derivative of (14.69) for t = 0 we get the following equation.

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 3A_2 e^{-3t} \rightarrow -12 = -2A_1 - 3A_2 \rightarrow 2A_1 + 3A_2 = 12 \tag{14.71}$$

A_1 and A_2 are obtained from (14.70) and (14.71).

$$A_1 = 12\,, \qquad A_2 = -4 \tag{14.72}$$

Finally, the requested voltage is given in (14.73).

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t}\,V\,, \qquad t > 0 \tag{14.73}$$

The requested current I obtained from (14.62).

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t}\,, \qquad \mathrm{t} > 0 \tag{14.74}$$

Thus,

$$i = 2 - 6e^{-2t} + 4e^{-3t}\,A\,, \qquad \mathrm{t} > 0 \tag{14.75}$$

Note that $i(0) = 0$ from (14.75) as is expected according to the initial condition in (14.56).

15. Laplace Transform in Circuit Analysis

15.1 Function transformation from time domain to s domain

Determine the Laplace transformation of each function given in Fig. 15.1:

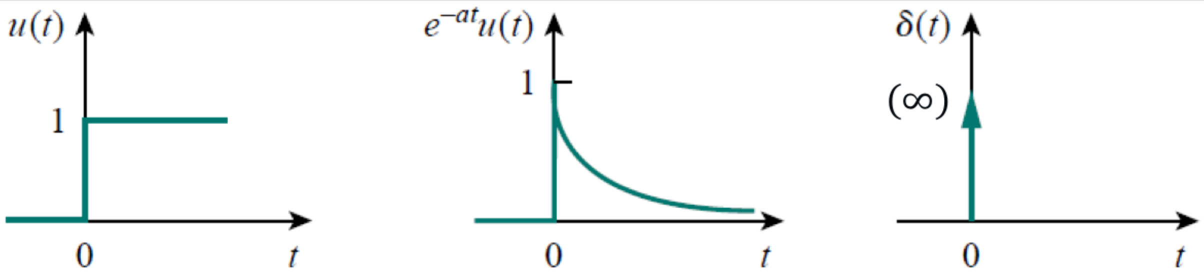

Figure 15.1 Time varying functions

Solution

By applying the integral transformation formula for unit step function we get the following result.

$$\mathcal{L}\{u(t)\} = \int_0^\infty 1\, e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s} \tag{15.1}$$

Integral transformation for the exponential function gives the following result.

$$\mathcal{L}\{e^{-at}u(t)\} = \int_0^\infty e^{-at}e^{-st}\, dt = -\frac{1}{s+a} e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a} \tag{15.2}$$

Finally, the unit pulse function in the Laplace domain is the following.

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)\, e^{-st}\, dt = e^{-0} = 1 \tag{15.3}$$

Note that the unit pulse function is the first derivative of the unit step function from which we see that the integral of the unit pulse 'around zero' is 1.

$$\delta(t) = \frac{du(t)}{dt} \rightarrow \int_{-0}^{+0} \delta(t)\, dt = 1 \tag{15.4}$$

An important practical application comes from (15.4), that is, the sampling or shifting property of the unit pulse, given in (15.5).

$$\int_{-0}^{+0} f(t) \cdot \delta(t)\, dt = f(0) \tag{15.5}$$

15.2 Laplace transform in circuit analysis 1

Find $v_o(t)$ in the circuit in Fig. 15.2, assuming zero initial conditions i.e. both, the capacitor and the inductor are energy-free at t = 0.

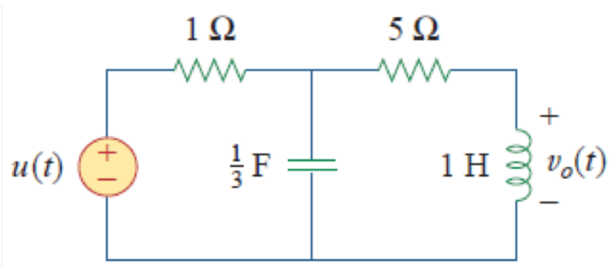

Figure 15.2 Circuit in time domain

Solution

The general steps that need to be followed in circuit analysis when applying the Laplace transform are the following. (1) Transform the circuit from the time domain to the *s* domain. (2) Solve the circuit using any circuit analysis technique (nodal, mesh analysis, source transformation, superposition, …) And finally, … (3) Take the inverse transformation of the solution to obtain the solution in the time domain.

The transformed circuit to Laplace domain is given in Fig. 15.3.

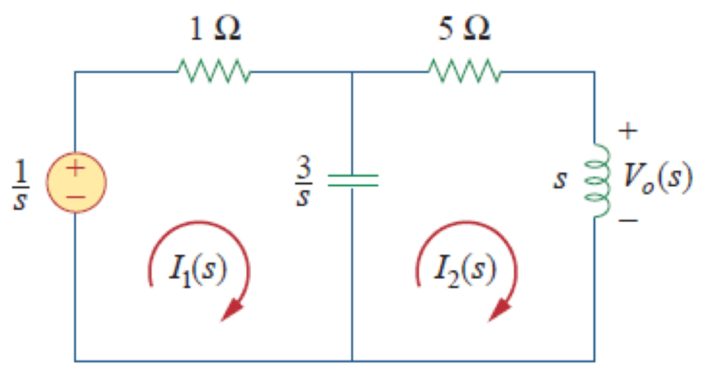

Figure 15.3 Circuit in Laplace (s) domain

Applying KVL for the loops in the circuit we get (15.6) and (15.7).

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2 \tag{15.6}$$

$$0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2 \rightarrow I_1 = \frac{1}{3}(s^2 + 5s + 3) I_2 \tag{15.7}$$

Substituting (15.7) into (15.6)

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) \frac{1}{3}(s^2 + 5s + 3) I_2 - \frac{3}{s} I_2 \tag{15.8}$$

From which,

$$3 = (s^3 + 8s^2 + 18s) I_2 \rightarrow I_2 = \frac{3}{s^3 + 8s^2 + 18s} \tag{15.9}$$

The requested voltage in s domain is

$$V_0(s) = s I_2 = \frac{3}{s^2 + 8s + 18} \tag{15.10}$$

This voltage has to be inverse Laplace transformed, so we need to change its formula to the 'appropriate' form to be found in the Laplace transform table. (Unless, we want to apply the inverse Laplace transformation method directly.) In the Laplace transform table we can find the formula as given in (15.11).

$$\mathcal{L}^{-1}\left\{\frac{\omega_d}{(s+a)^2+\omega_d^2}\right\} = e^{-at}\cdot\sin(\omega_d t) \tag{15.11}$$

Thus, we modify (15.10) as in (15.12).

$$V_0(s) = \frac{3}{s^2+8s+18} = \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2+\sqrt{2}^2} \tag{15.12}$$

By substituting the obtained parameters into (15.11) the requested voltage in the time domain is given in (15.13).

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = \frac{3}{\sqrt{2}}e^{-4t}\sin\left(\sqrt{2}\,t\right)u(t)\,V,\ t\geq 0 \tag{15.13}$$

15.3 Laplace transform in circuit analysis 2

Obtain $v_o(t)$ in the circuit, shown in Fig.15.4, assuming $v_o(0)$ = 5 V.

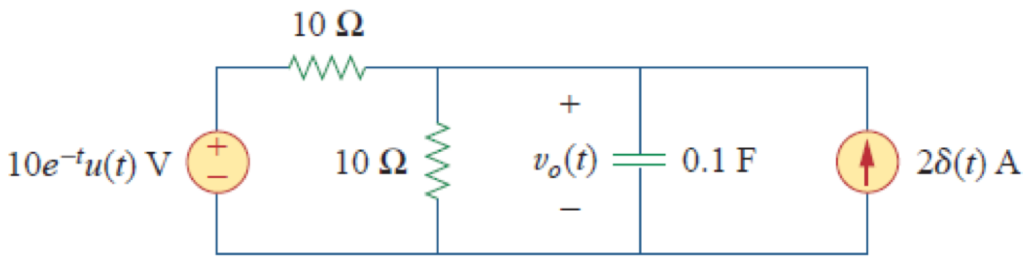

Figure 15.4 Circuit with initial condition

Solution

Circuit, transformed to s domain, is shown in Fig. 15.5.

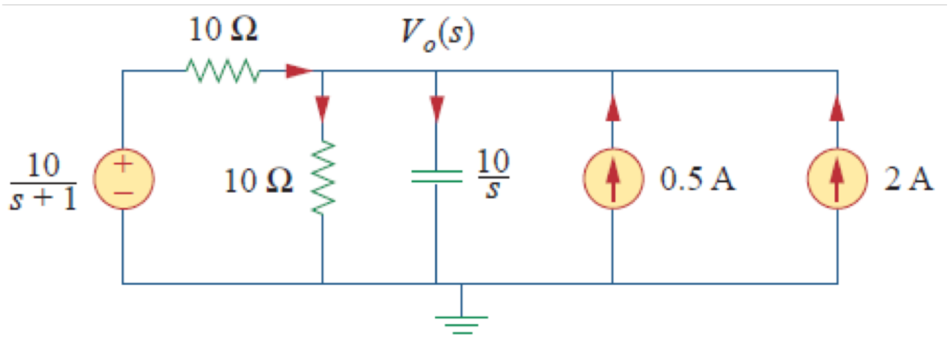

Figure 15.5 Circuit in Laplace domain

Voltage and current sources are transformed according to (15.2) and (15.4) or applied (any) Laplace transform table. The result is given in (15.14)

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}\,, \qquad \mathcal{L}\{\delta(t)\} = 1 \tag{15.14}$$

From the initial voltage across the capacitor the calculated initial condition is

$$C \cdot v_0(0) = 0.1 \cdot 5 = 0.5\,A \tag{15.15}$$

Applying nodal analysis, we can write the following equation.

$$\frac{{}^{10}\!/\!{}_{(s+1)} - V_0}{10} + 2 + 0.5 = \frac{V_0}{10} + \frac{V_0}{{}^{10}\!/\!{}_{s}} \tag{15.16}$$

With sorting this equation, we can express V_0

$$\frac{1}{s+1} + 2.5 = \frac{2V_0}{10} + \frac{sV_0}{10} = \frac{1}{10}V_0(s+2) \tag{15.17}$$

In shorter form

$$\frac{10}{s+1} + 25 = V_0(s+2) \tag{15.18}$$

Thus,

$$V_0(s) = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \tag{15.19}$$

We have to determine A and B parameters. A is derived from (15.20).

$$A = (s+1) \cdot V_0(s)|_{s=-1} = \left.\frac{25s+35}{(s+2)}\right|_{s=-1} = \frac{10}{1} = 10 \tag{15.20}$$

And B is derived from (15.21),

$$B = (s+2) \cdot V_0(s)|_{s=-2} = \left.\frac{25s+35}{(s+1)}\right|_{s=-2} = \frac{-15}{-1} = 15 \tag{15.21}$$

Thus, V_0 voltage in s domain has the following expression.

$$V_0(s) = \frac{10}{(s+1)} + \frac{15}{(s+2)} \tag{15.22}$$

The inverse Laplace transformation of (15.22) gives the following:

$$v_0(t) = \mathcal{L}^{-1}\{V_0(s)\} = (10 \cdot e^{-t} + 15 \cdot e^{-2t}) \cdot u(t)\,V \tag{15.23}$$

15.4 Circuit with a switch

The switch in circuit, shown in Fig. 15.6, moves from position *a* to position *b* at t = 0. Find i(t) for t > 0.

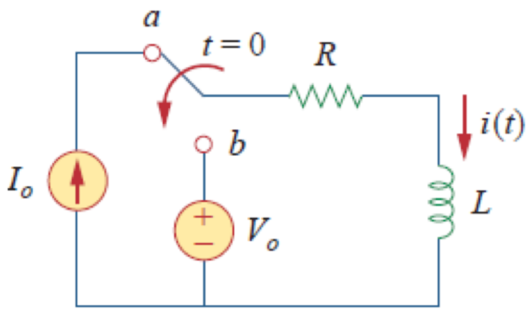

Figure 15.5 Circuit with a switch

Solution

First of all, the switch in position ‘a’ sets the initial condition only. The circuit we have to analyse is in the switch position ‘b’, that is initially a source free circuit.

When the switch is in position ‘a’ the initial condition is given in (15.24)

$$i(0) = I_0 \tag{15.24}$$

The circuit to be examined is given in s domain as shown in Fig. 15.7.

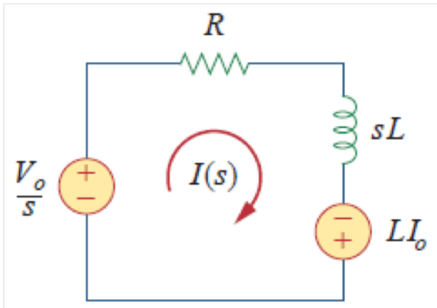

Figure 15.7 Circuit transformed to s domain

By applying KVL for the single loop we can write (15.25).

$$I(s)(R + sL) - L \cdot I_0 - \frac{V_0}{s} = 0 \tag{15.25}$$

Expressing *I(s)* from the equation (15.25)

$$I(s) = L \cdot \frac{I_0}{R + sL} + \frac{V_0}{s(R + sL)} = \frac{I_0}{s + {}^{R}/_{L}} + \frac{{}^{V_0}/_{L}}{s\left(s + {}^{R}/_{L}\right)} \tag{15.26}$$

After mathematical transformation it can be written as

$$I(s) = \frac{I_0}{s + {}^{R}/_{L}} + \frac{{}^{V_0}/_{R}}{s} - \frac{{}^{V_0}/_{R}}{s + {}^{R}/_{L}} = \frac{I_0 - {}^{V_0}/_{R}}{s + {}^{R}/_{L}} + \frac{{}^{V_0}/_{R}}{s} \tag{15.27}$$

From which the requested current in the time domain is obtained by applying the inverse Laplace transform.

$$i(t) = \left(I_0 - \frac{V_0}{R}\right) e^{-t/\tau} + \frac{V_0}{R}\,, \qquad t \geq 0 \tag{15.28}$$

15.5 Circuit analysis applying the superposition principle

Use the superposition theorem to find the capacitor voltage for the circuit in Fig. 15.8. The initial conditions are the following $i_L(0) = -1$ A , $v_C(0) = +5$ V

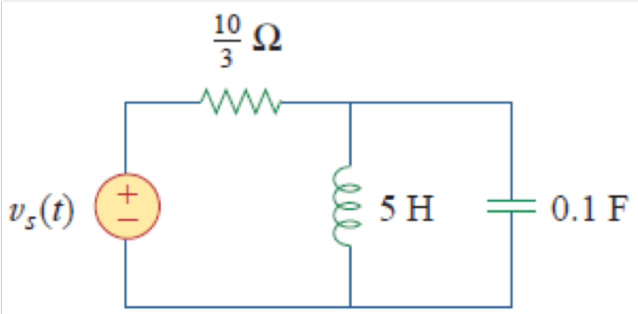

Figure 15.8 Circuit in time domain

Solution

The circuit has only one source, but even so, we can use the superposition principle in our calculation as the initial conditions can be considered as additional sources. By transforming the circuit from the time domain to the s domain we get the circuit, given in Fig. 15.9.

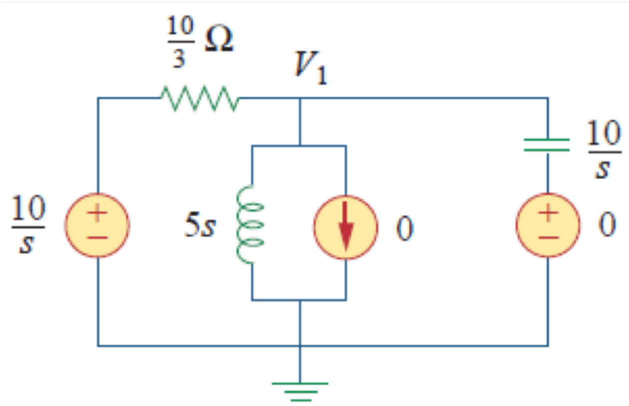

Figure 15.9 Circuit in s domain

By using the superposition principle (the network is linear), we can do the calculation in three steps. Let's have the voltage source in remain to start. By applying nodal analysis the following nodal equation is given.

$$\frac{V_1 - {}^{10}/_{s}}{{}^{10}/_{3}} + \frac{V_1 - 0}{5s} + 0 + \frac{V_1 - 0}{{}^{10}/_{s}} = 0 \tag{15.29}$$

Sorting the equation, we get the following.

$$0.1\left(s + 3 + \frac{2}{s}\right) \cdot V_1 = \frac{3}{s} \tag{15.30}$$

From which,

$$(s^2 + 3s + 2) \cdot V_1 = 30 \tag{15.31}$$

Expressing V_1 from (15.31) the result is

$$V_1 = \frac{30}{(s+1)(s+2)} = \frac{30}{s+1} - \frac{30}{s+2} \tag{15.32}$$

By applying inverse Laplace transform we get the voltage across the capacitor, caused by the single voltage source

$$v_1(t) = (30 \cdot e^{-t} - 30 \cdot e^{-2t}) \cdot u(t)\ V \tag{15.33}$$

The next step is to calculate the voltage across the capacitor due to the current source, that is, from the initial condition of the inductor. The circuit for this calculation is given in Fig. 15.10.

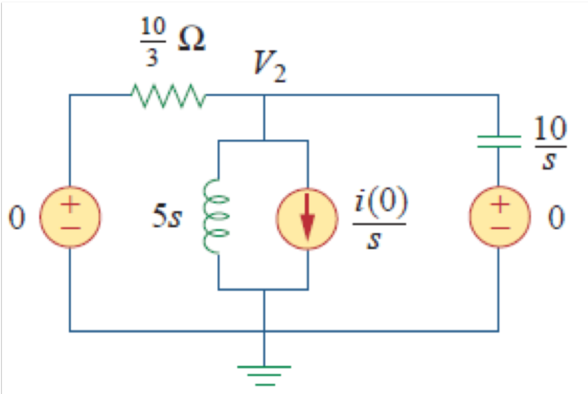

Figure 15.10 Circuit with the current source only

The nodal equation in this case is the following.

$$\frac{V_2 - 0}{10/3} + \frac{V_2 - 0}{5s} - \frac{1}{s} + \frac{V_2 - 0}{10/s} = 0 \tag{15.34}$$

By sorting the equation, we get (15.35),

$$0.1\left(s + 3 + \frac{2}{s}\right) \cdot V_2 = \frac{1}{s} \tag{15.35}$$

from which,

$$(s^2 + 3s + 2) \cdot V_2 = 30 \tag{15.36}$$

By expressing V_2 the result is

$$V_2 = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2} \tag{15.37}$$

Through applying inverse Laplace transform for (15.37) we get the voltage across the capacitor, caused by the initial condition of the inductor.

$$v_2(t) = (10 \cdot e^{-t} - 10 \cdot e^{-2t}) \cdot u(t)\ V \tag{15.38}$$

The last (third) source to be considered is the voltage from the initial condition of the capacitor. The circuit for this calculation is given in Fig. 15.11.

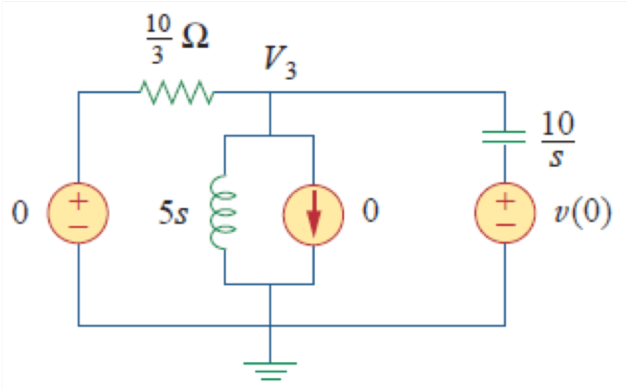

Figure 15.11 Circuit with capacitors initial voltage

The nodal equation in this case is given in (15.39).

$$\frac{V_3 - 0}{10/_3} + \frac{V_3 - 0}{5s} - 0 + \frac{V_3 - 5/_s}{10/_s} = 0 \tag{15.39}$$

After sorting we get (15.40),

$$0.1\left(s + 3 + \frac{2}{s}\right) \cdot V_3 = 0.5 \tag{15.40}$$

Expressing V_3,

$$V_3 = \frac{5s}{(s+1)(s+2)} = \frac{-5}{s+1} + \frac{10}{s+2} \tag{15.41}$$

By Applying inverse Laplace transform for (15.41) we get the voltage across the capacitor, a result of its initial condition.

$$v_3(t) = (-5 \cdot e^{-t} + 10 \cdot e^{-2t}) \cdot u(t)\ V \tag{15.42}$$

According to the superposition principle the final result is given in (15.43)

$$v(t) = v_1(t) + v_2(t) + v_3(t) \tag{15.43}$$

By substituting (15.33), (15.38), and (15.42) into (15.43) the requested voltage is

$$v(t) = \{(30 + 10 - 5) \cdot e^{-t} + (-30 - 10 + 10) \cdot e^{-2t}\} \cdot u(t) \tag{15.44}$$

Or, after simplifying the equation

$$v(t) = \{35 \cdot e^{-t} - 30 \cdot e^{-2t}\} \cdot u(t) \tag{15.45}$$

16. Fourier Transform in Circuit Analysis

16.1 Function transformation from time domain to frequency domain

Find the Fourier transform of the following functions (a) $\delta(t-t_0)$, (b) $e^{j\omega_0 t}$, (c) $\cos\omega_0 t$.

Solution (a)

Using the definition of the Fourier integral transform and applying the shifting property of unit pulse function we can write (16.1).

$$F(\omega)=\mathcal{F}[\delta(t-t_0)]=\int_{-\infty}^{\infty}\delta(t-t_0)e^{-j\omega t}dt=e^{-j\omega t_0} \tag{16.1}$$

The shifting property in the special case of $t_0=0$ (16.1) gives the result of (16.2).

$$\rightarrow \mathcal{F}[\delta(t)]=1 \tag{16.2}$$

Solution (b)

For this solution we simply use any of the Fourier transform tables. This table is given in Fig. 16.1 where, in the middle row, we can find the result as given in (16.3).

$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\mathrm{sgn}\,(t)$	$\dfrac{2}{j\omega}$

Figure 16.1 Fourier transform table (part)

$$F(\omega)=\mathcal{F}\left[e^{j\omega_0 t}\right]=2\pi\delta(\omega-\omega_0) \tag{16.3}$$

Solution (c)

To find the Fourier transform of the $\cos\omega_0 t$ function we apply Euler's formula to get the result given in (16.4).

$$F(\omega)=\mathcal{F}[\cos\omega_0 t]=\mathcal{F}\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right] \tag{16.4}$$

According to the linearity property of Fourier transform it can be written as in (16.5).

$$\mathcal{F}\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]=\frac{1}{2}\mathcal{F}\left[e^{j\omega_0 t}\right]+\frac{1}{2}\mathcal{F}\left[e^{-j\omega_0 t}\right] \tag{16.5}$$

By substituting (16.3) into (16.5) we get the following result.

$$\frac{1}{2}\mathcal{F}\left[e^{j\omega_0 t}\right]+\frac{1}{2}\mathcal{F}\left[e^{-j\omega_0 t}\right]=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0) \tag{16.6}$$

The Fourier transform of the cosine function is also drawn in Fig. 16.2.

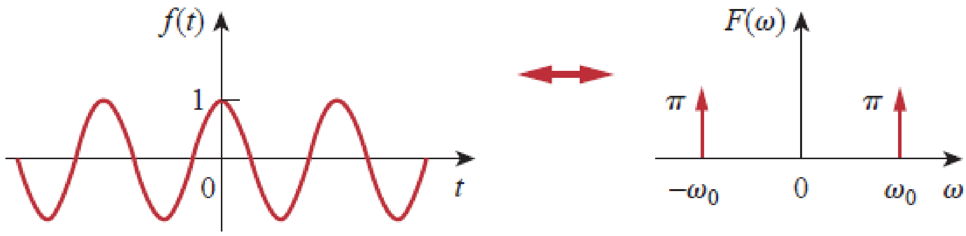

Figure 16.2 Fourier transform of the cosine function

16.2 The sinus cardinalis (sinc) function

Find the Fourier transform of the single (nonperiodic) rectangular pulse shown in Fig. 16.3

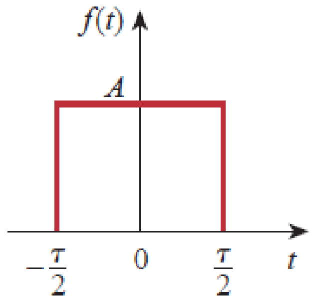

Figure 16.3 Nonperiodic rectangular function

Solution

Substituting the single pulse function into the equation of the Fourier transform

$$F(\omega) = \int_{-\tau/2}^{\tau/2} Ae^{-j\omega t}dt = -\frac{A}{j\omega}\bigg|_{-\tau/2}^{+\tau/2} = \frac{2A}{\omega}\left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right) \tag{16.7}$$

Applying Euler's formula, the result is given in (16.8) where the (sin x)/x is *renamed* to sinc (x).

$$F(\omega) = A\tau\frac{\sin \omega\tau/2}{\omega\tau/2} = A\tau \,\mathrm{sinc}\, \omega\tau/2 \tag{16.8}$$

Apply, for example, the following parameters.

$$A = 10, \tau = 2 \rightarrow F(\omega) = 20 \,\mathrm{sinc}\, \omega \tag{16.9}$$

The equation in (16.9) is drawn in Fig. 16.4.

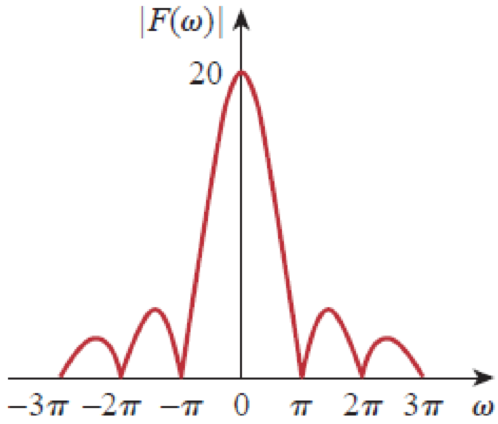

Figure 16.4 The sinc function of (16.9)

Please note this solution is ‘similar’ to the discrete spectrum i.e. the Fourier series of a periodic pulse train, that we previously calculated in Chapter 11. For a comparison of the continuous spectrum of a single rectangular pulse and the discrete spectrum of the periodic pulse train we give the periodic pulse train in Fig. 16.5 and its Fourier series in Fig. 16.6.

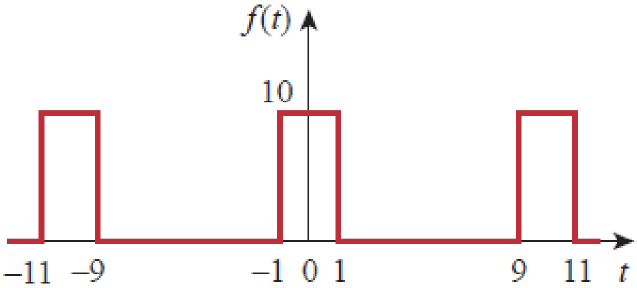

Figure 16.5 Periodic pulse train

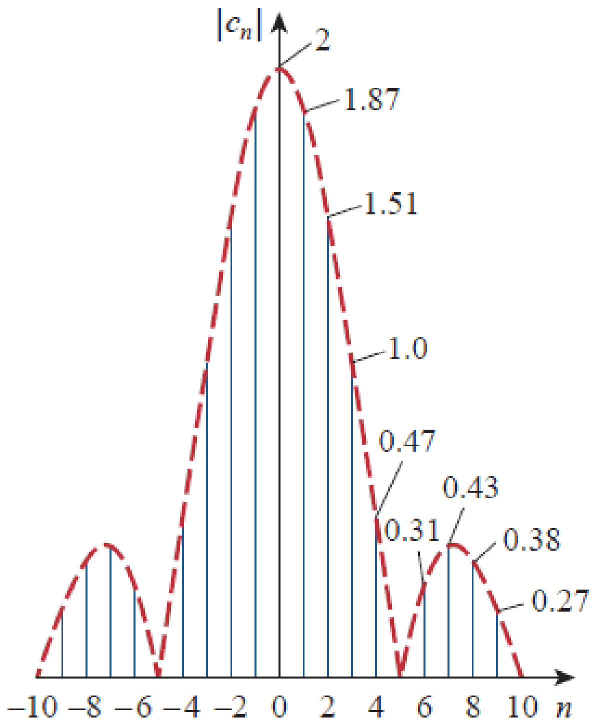

Figure 16.6 Discrete spectrum of the periodic pulse train

16.3 Circuit application 1

Find $v_0(t)$ in the circuit shown in Fig. 16.7 for $v_i(t) = 2e^{-3t}u(t)\ V$

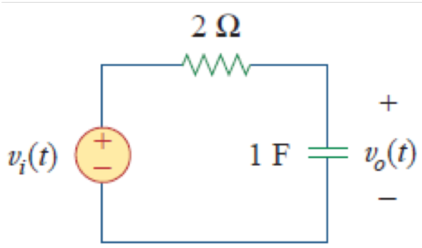

Figure 16.7 Circuit for finding *vO(t)* voltage

Solution

The Fourier transform of the source voltage is given in (16.10).

$$V_i(\omega) = \frac{2}{3 + j\omega} \tag{16.10}$$

The transfer function of the circuit is given in (16.11)

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1/j\omega}{2 + 1/j\omega} = \frac{1}{1 + j2\omega} \tag{1611}$$

thus, the output voltage in the Laplace domain is

$$V_o(\omega) = V_i(\omega)H(\omega) = \frac{2}{(3 + j\omega)(1 + j2\omega)} = \frac{1}{(3 + j\omega)(0.5 + j\omega)} \tag{16.12}$$

To be able to inverse the Laplace transform of the function, given in (16.12) the partial fractions have to be determined. The result is in (16.13).

$$V_o(\omega) = \frac{-0.4}{3 + j\omega} + \frac{0.4}{0.5 + j\omega} \tag{16.13}$$

Two parts of (16.13) can be transformed using the inverse Laplace giving the result in (16.14).

$$\mathcal{F}^{-1}[V_o(\omega)] = v_0(t) = 0.4(e^{-0.5t} - e^{-3t})u(t) \tag{16.14}$$

16.4 Circuit application 2

Find $i_0(t)$ in the circuit shown in Fig. 16.8 for $i_s(t) = 10 \sin 2t\ A$.

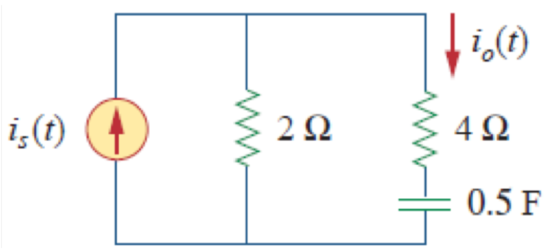

Figure 16.8 Circuit for finding *iO(t)* current

Solution

Because the current source is a sinusoidal source, the calculation could also be done in the phasor domain. Now, we apply the Laplace transformation method that naturally gives the same result. The Laplace transform of the source current is given in (16.15).

$$I_S(\omega) = j\pi 10[\delta(\omega+2) - \delta(\omega-2)] \tag{16.15}$$

The transfer function of the circuit is

$$H(\omega) = \frac{I_o(\omega)}{I_S(\omega)} = \frac{2}{2\ +4+2/j\omega} = \frac{j\omega}{1+j\omega 3} \tag{16.16}$$

Thus, the output current in Laplace domain is

$$I_0(\omega) = H(\omega)I_S(\omega) = \frac{10\pi\omega[\delta(\omega-2) - \delta(\omega+2)]}{1+j\omega 3} \tag{16.17}$$

The output current in time domain is given by the inverse Laplace transformation.

$$i_0(t) = \mathcal{F}^{-1}[I_o(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{10\pi\omega[\delta(\omega-2) - \delta(\omega+2)]}{1+j\omega 3} e^{j\omega t} d\omega \tag{16.18}$$

Applying the shifting property of the unit pulse function we can write (16.19).

$$\int_{-\infty}^{\infty} \delta(\omega-\omega_0) f(\omega) d\omega = f(\omega_0) \tag{16.19}$$

From (16.18) and (16.19) we can write the following.

$$i_0(t) = \frac{10\pi}{2\pi}\left[\frac{2}{1+j6}e^{j2t} - \frac{-2}{1-j6}e^{-j2t}\right] \tag{16.20}$$

Once we rewrite (1+j6) and (1-j6) in polar form we obtain (16.21).

$$i_0(t) = 10\left[\frac{e^{j2t}}{6.082\ e^{j80.54°}} - \frac{e^{-j2t}}{6.082\ e^{-j80.54°}}\right] \tag{16.21}$$

Thus,

$$i_0(t) = 1.644\left[e^{j(2t-80.54°)} + e^{-j(2t-80.54°)}\right] \tag{16.22}$$

By applying the Euler formula for (16.22) the response current in the circuit is given in (16.23).

$$i_0(t) = 3.288\cos(2t - 80.54°)\ A \tag{16.23}$$

The response function is a harmonic function as was expected because of the harmonic excitation in the circuit.

16.5 Parseval’s theorem

The voltage across a 10-Ω resistor is $v(t) = 5e^{-3t}u(t)\ V$. Find the total energy dissipated in the resistor.

Solution 1

We need to first calculate the total dissipated energy in the time domain.

The total energy dissipated by the 10-Ω resistor is given from (16.24).

$$W_{10\Omega} = \frac{1}{10}\int_{-\infty}^{\infty} v^2(t)\, dt = 0.1\int_{0}^{\infty} 25\, e^{-6t} dt = 2.5\left.\frac{e^{-6t}}{-6}\right|_0^\infty \qquad (16.24)$$

When solving this equation it gives (16.25).

$$W_{10\Omega} = 2.5\left.\frac{e^{-6t}}{-6}\right|_0^\infty = \frac{2.5}{6} = 416.7\ mJ \qquad (16.25)$$

Solution 2

Now we calculate the total dissipated energy in the frequency domain by the application of Parseval's theorem. The voltage across the resistor in the frequency domain is the Fourier transform of the given voltage, and the square of the absolute value is given in (16.26).

$$V(\omega) = \frac{5}{3+j\omega} \rightarrow |V(\omega)|^2 = V(\omega)V(\omega)^* = \frac{25}{9+\omega^2} \qquad (16.26)$$

According to Parseval's theorem the total energy dissipated by the 10-Ω resistor is

$$W_{10\Omega} = \frac{0.1}{2\pi}\int_{-\infty}^{\infty} |V(\omega)|^2 d\omega = \frac{0.1}{\pi}\int_{0}^{\infty} \frac{25}{9+\omega^2} d\omega \qquad (16.27)$$

From an integral table we have (16.28) to be applied for (16.27).

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} tan^{-1}\frac{x}{a} \qquad (16.28)$$

Thus, the result for total dissipated energy is given in (16.29) which is the same as calculated in (16.25).

$$W_{10\Omega} = \frac{2.5}{\pi}\left.\left(\frac{1}{3}\tan^{-1}\frac{\omega}{3}\right)\right|_0^\infty = \frac{2.5}{\pi}\cdot\frac{1}{3}\cdot\frac{\pi}{2} = \frac{2.5}{6} = 416.7\ mJ \qquad (16.29)$$

16.6 Amplitude modulation (AM)

A music signal has frequency components from 15 Hz to 30 kHz and this signal is used to amplitude modulate a 1.2-MHz carrier. Find the range of frequencies for the lower and upper sidebands.

Solution

The frequency spectrum with lower sideband (LSB) and upper sideband (USB) of amplitude modulation is given in Fig. 16.9.

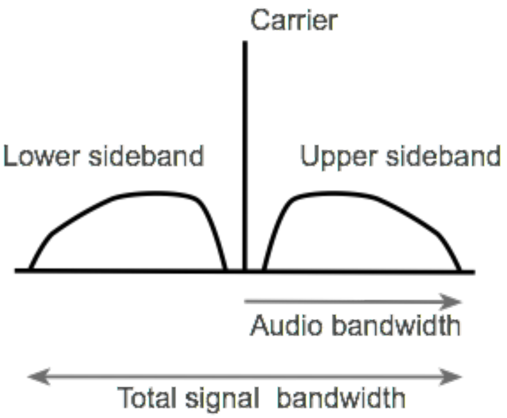

Figure 16.9 Frequency spectrum of amplitude modulation

The lowest cut-off frequency of the applied bandwidth is given in (16.30).

$$f_{LSB1} = 1{,}200{,}000 - 30{,}000 = 1{,}1700{,}000\ Hz \tag{16.30}$$

Another cut-off frequency point of the lower sideband is in (16.31).

$$f_{LSB2} = 1{,}200{,}000 - 15 = 1{,}199{,}995\ Hz \tag{16.31}$$

The next point over the carrier frequency is the lower cut-off frequency of the upper sideband, that is shown in (16.32).

$$f_{USB3} = 1{,}200{,}000 + 15 = 1{,}200{,}015\ Hz \tag{16.32}$$

Finally, the upper cut-off frequency of the upper sideband is shown in (16.33).

$$f_{USB4} = 1{,}200{,}000 + 30{,}000 = 1{,}230{,}000\ Hz \tag{16.33}$$

Thus, the total bandwidth used by the given AM signal is between 1.17 MHz and 1.23 MHz. The necessary bandwidth for the AM transmission of the 15Hz – 30 kHz audio band is two times higher i.e. 60 kHz in this case, because of the double sideband modulation.

16.7 Nyquist–Shannon sampling theorem

A telephone signal with a cut-off frequency of 5 kHz is sampled at a rate 60 percent higher than the minimum allowed rate. Find the sampling rate.

Solution

The sampling rate according to Nyquist-Shannon theorem is two times higher than the upper cut-off frequency. That is the minimum allowed sampling rate or Nyquist rate, calculated in (16.34).

$$f_{S\,min} = 2 \cdot f_{cut-off} = 2 \cdot 5 = 10\ kHz \tag{16.34}$$

Because the telephone signal with the given cut-off frequency is sampled at a rate 60 percent higher than the minimum allowed rate, the applied sampling rate is calculated in (16.35).

$$f_S = 1.6 \cdot f_{S\,min} = 1.6 \cdot 10 = 16\ kHz \tag{16.35}$$

17. Sources and Recommended Additional Materials

17.1 In English

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   GlobeEdit, 2016, ISBN:978-3-330-71341-3
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